Chapter I Gromov's Nonsqueezing Thm

§ I.3 IDEA of the Proof

Notation: $B_r^{2n} := \{ z \in \mathbb{C}^n : |z| < r \}$. $B^{2n} := B_1^{2n}$

$$Z := Z^{2n} := \{ z \in \mathbb{C}^n : |z_1| < 1 \} = B^2 \times \mathbb{C}^{n-1}$$

FACT: $\forall r$, we can volume-presered embedd B_r^{2n} into Z.

$\Leftarrow$ $\Phi_S : \mathbb{C}^n \to \mathbb{C}^n$, $(z_1, \cdots, z_n) \longmapsto (\frac{z_1}{S}, sz_2, \cdots, z_n)$, if $S \geq r$

then $\Phi_S(B_r) \subset Z$, and for $\omega = \sum_{i=1}^n dx^i \wedge dy^i$.

$$\Phi_S^* \omega = \frac{1}{S^2} dx^1 \wedge dy^1 + s^2 dx^2 \wedge dy^2 + \cdots + dx^n \wedge dy^n$$

$\Rightarrow \Phi_S^* \omega^n = (\Phi_S^* \omega)^n = \mathrm{vol}_{\mathbb{C}^n} = n! \, dx^1 \wedge \cdots \wedge dy^n$

$\rightsquigarrow$ Φ_S is volume preserving, but not "symplectic embedding".

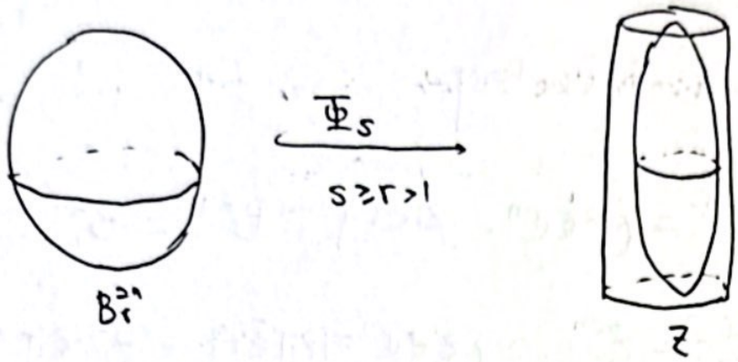
Def: If (M, ω_M) and (N, ω_N) are two symplectic mflds, with embedding $i : M \hookrightarrow N$. we call this embedding is symplectic if $i^* \omega_N = \omega_M$.

Actually, we have

Thm: (Gromov - Nonsqueezing theorem)

The ball (B_r, ω) embedds symplectically into (Z, ω) iff $r \leq 1$.

Rmk: From the thm above, we can see the "rigidity" of symplectic.



IDEA OF THE PROOF:

Step 1: Suppose we have a symplectic embedding

$$\Phi : B_{r_0} \longrightarrow Z$$

recall this is an open ball

Our aim is to show that $r_0 \leq 1$.

We consider the closed ball $\mathbb{D}_r := \overline{B_r}$, $r < r_0$, and write $\Phi^{-1} : \Phi(\mathbb{D}_r) \to \mathbb{D}_r$. it suffices to for $\forall r$, $r \leq 1$.

Let $\omega, J_0, \langle \cdot, \cdot \rangle_{J_0}$ be the standard str. on $\mathbb{C}^n = \mathbb{R}^{2n}$.

We can construct an almost complex str. J on $\Phi(\mathbb{D}_r)$ as

$$J := \Phi_* \circ J_0 \circ \Phi_*^{-1}$$

– FACT: J is an almost complex str. compatible with ω.

Pf:
$$\begin{aligned}
\langle X, Y \rangle_J &:= \omega(X, JY) \\
&= \omega(X, \Phi_* \circ J_0 \circ \Phi_*^{-1} Y) \\
&= \omega(\Phi_*^{-1} X, J_0 \circ \Phi_*^{-1} Y) \\
&= \langle \Phi_*^{-1} X, \Phi_*^{-1} Y \rangle_{J_0}
\end{aligned}$$

Φ (hence Φ^{-1}) is a symplectomorphism from $\mathbb{D}^r \leftrightarrows \Phi(\mathbb{D}^r)$

$\square$

Step 2: $\boxed{\text{(non-trivial) FACT:}}$ Main Part. J can be extended to an almost complex str. compatible with ω on $\mathbb{R}^{2n}$, and conicides with J_0 outside a small nbhd of $\Phi(\mathbb{D}_r)$.

from the FACT $\Longrightarrow$ we can establish the existence of J-holomorphic Disc.

i.e. a smooth map $\hat{u}:(\mathbb{D}, \partial\mathbb{D}) \to (Z, \partial Z)$ with

(D1) $\hat{u}(0) = \Phi(0)$

(D2) $\int_{\mathbb{D}} \hat{u}^*\omega = \pi$ — 「this is called the symplectic energy of $\hat{u}$」

(D3) $\hat{u}_x + J(u)(\hat{u}_y) = 0.$ — 「u_x is a vector. and can be identified with the tangent vector. $J(u) = J_{u(z)}$.」

Def: Let (M, J) be an almost complex mfld. and $U \subseteq \mathbb{C}$ be some open subset. A smooth map $u: U \to M$ is called a J-holomorphic curve w.r.t to J. if $\hat{u}_x + J(u)(\hat{u}_y) = 0$.

Rmk: More generally. take j as the complex str. $\hat{u}_x + J(u)(\hat{u}_y) = 0 \Longleftrightarrow \hat{U}_* \circ j = J \circ \hat{U}_*$

Pf: For $\{\partial x, \partial y\} = T\hat{u}$. $\hat{U}_* \circ j(\partial x) = J \circ \hat{U}_*(\partial x) \Rightarrow \hat{U}_*(\partial y) = J(\hat{u}_x)$

$\Rightarrow \hat{u}_x = -J(\hat{u}_y) \Rightarrow \hat{u}_x + J(\hat{u}_y) = 0$. the opposite side is same and easy.

□

Exercise: Let $\varphi: V \to U$ be a biholomophism. $u: U \to (M, J)$ is a J-holomorphic cure $\Longrightarrow$ $u \circ \varphi$ is also a J-holomorphic curve. i.e. conformal reparametrisations of J-holomophic curves are still J-holomophic.

Pf: Let $U_{(x,y)}$ and $V_{(s,t)}$, then $\varphi = (x(s,t), y(s,t))$

$$\Rightarrow (u\circ\varphi)_s = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial x}{\partial s} = \frac{\partial y}{\partial t}, \quad \frac{\partial x}{\partial t} = -\frac{\partial y}{\partial s}$$

$$(u\circ\varphi)_t = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \underbrace{\frac{\partial u}{\partial y}}_{\text{vector}}\underbrace{\frac{\partial y}{\partial t}}_{\text{number}},$$

$$\Rightarrow (u\circ\varphi)_s + J\big((u\circ\varphi)_t\big)$$

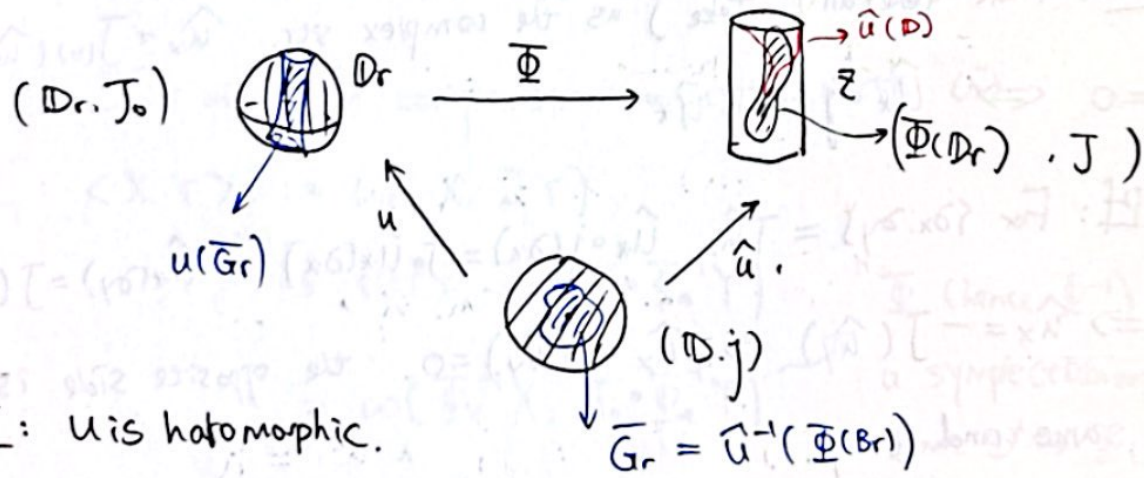
$$= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + J\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= \frac{\partial x}{\partial s}\big(\underbrace{u_x + J(u_y)}_{=0}\big) + \frac{\partial y}{\partial s}\big(\underbrace{u_y - J(u_x)}_{=0}\big) = 0.$$

$\square$.

Step 3: Consider $G_r := \hat{u}^{-1}(\overline{\Phi(B_r)}) \subseteq \mathbb{D}$. then define

$$u := \Phi^{-1} \circ \hat{u}|_{\overline{G_r}} : \overline{G_r} \to \mathbb{D}_r$$



FACT: u is holomophic.

Pf: Recall $J = \Phi_* \circ J_0 \circ \Phi_*^{-1}$. then we have

$$u_* \circ j = \Phi_*^{-1} \circ \hat{u}_* \circ j = \Phi_*^{-1} \circ J \circ \hat{u}_*$$
$$= J_0 \circ \Phi_*^{-1} \circ \hat{u}_* = J_0 \circ u_*$$

$\Rightarrow$ ~~u_*~~ u is J_0-holomorphic. i.e. holomorphic since J_0 is standard.

$\square$

<u>Claim</u>: $\overline{G_r} \subseteq \mathbb{D}^\circ$, $u(\partial G_r) \subset \partial \mathbb{D}_r$.

<u>Hint</u>: This is from basic set topology.

<u>Step4</u>: We apply the monotonicity lemma. i.e. under this situation, we have

$$\pi r^2 \leq \int_{\overline{G_r}} u^*\omega$$

Note that $\hat{u} = \Phi \circ u$. since Φ is sympectic $\Rightarrow$

$$\hat{u}^*\omega = (\Phi \circ u)^*\omega = u^*\Phi^*\omega = u^*\omega$$

$$\Rightarrow \int_{\overline{G_r}} u^*\omega = \int_{\overline{G_r}} \hat{u}^*\omega.$$

<u>Note that</u> $\hat{u}^*\omega$ is non-negative on $\mathbb{D}$. i.e.

$$\hat{u}^*\omega(\partial_x, \partial_y) = \omega(\hat{u}_x, \hat{u}_y)$$
$$= \omega(\hat{u}_x, J(\hat{u}_x)) = \langle \hat{u}_x, \hat{u}_y \rangle_J \geq 0$$

$$\Rightarrow \int_{\overline{G_r}} u^*\omega = \int_{\overline{G_r}} \hat{u}^*\omega \leq \int_{\mathbb{D}} \hat{u}^*\omega = \pi$$ By (D_3)

$\Rightarrow r \leq 1$. as desired.

##

§ I.4 The monotonicity Lemma.

Thm: If $G \subseteq \mathbb{C}$ be a domain. $0 \in G$. and $u: (G, 0) \to (\mathbb{C}^n, 0)$ is a proper holomorphic map. then for any $r > 0$. we have

$$\int_{u^{-1}(B_r)} u^*\omega \geq \pi r^2 .$$

- The proof is rather easy. main tool is to use Isoperimetric inequalities. We omit here. Ref [Geige. Kai § I.4 – I.6]

##

§ I.7 Extension of J from $\Phi(\mathbb{D}_r)$ to $\mathbb{C}^n$.

Thm: There is an almost complex structure $\hat{J}$ on $\mathbb{R}^{2n}$, compatible with ω. that coincides with J on $\Phi(\mathbb{D}_r)$, and with J_0 on $\mathbb{R}^{2n} \setminus \Phi(B_{r+\varepsilon})$, where ε is small enough s.t. $r+\varepsilon < r_0$.

- The proof is by directly construction. with the help of cut-off function and linear algebra. Ref [G.K. § I.7]

Now we can start the really interesting part of the proof

⟹ the existence of J-holomorphic Discs.

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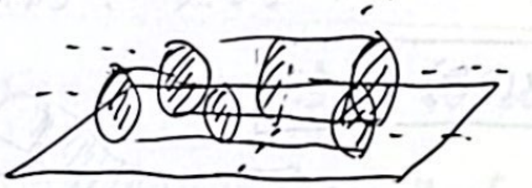
§ I.8 A Moduli space of J- holomorphic discs.

To find or prove the existence of J-holomorphic curves, we always try to consider the Moduli space, then study the property of Moduli space, we can know the desired target is exists.

For Cylinder $Z = B \times \mathbb{C}^{n-1}$, with coordinate $(\omega, x+iy)$.

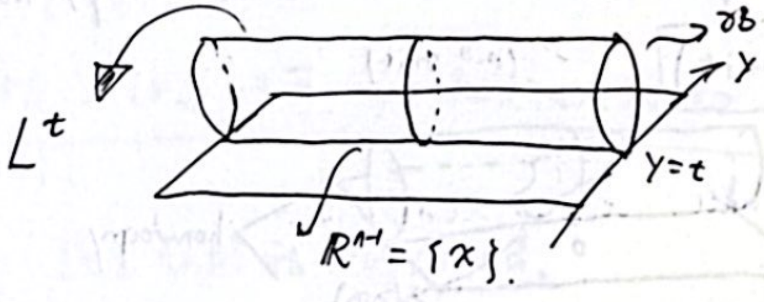
$\omega \in B \subseteq \mathbb{C}$, $|\omega|=1$. $x, y \in \mathbb{R}^{n-1}$. $x=(x_2, \cdots, x_n)$, $y=(y_2, \cdots, y_n)$.

One can view Z in $n=2$ as a disk bundle over $\mathbb{C}^{n-1}$



Hence the boundary of Z, $\partial Z = \partial B \times \mathbb{C}^{n-1}$ has a family of n-dim cylinders, $t \in \mathbb{R}^{n-1}$

$$L^t := \partial B \times \mathbb{R}^{n-1} \times \{y=t\}$$



FACT: $\dim_{\mathbb{R}} L^t = n$. and actually L^t is a Lagranigian submfld of $(\mathbb{R}^{2n}, \omega)$.

PROOF: $i: L^t \hookrightarrow \mathbb{R}^{2n}$. by $(\cos\theta, \sin\theta, x_2, t_2, \cdots, x_n, t_n)$ t_i are consts.

$\Rightarrow i^*\omega = d\cos\theta \wedge d\sin\theta + \sum_{i=2}^{n} dx_i \wedge 0 = 0$ since $\dim L^t = \frac{1}{2}\dim \mathbb{R}^{2n}$

$\Rightarrow L^t$ is Langramian. $\square$

Now we consider the Moduli space $\mathcal{M}$ as:

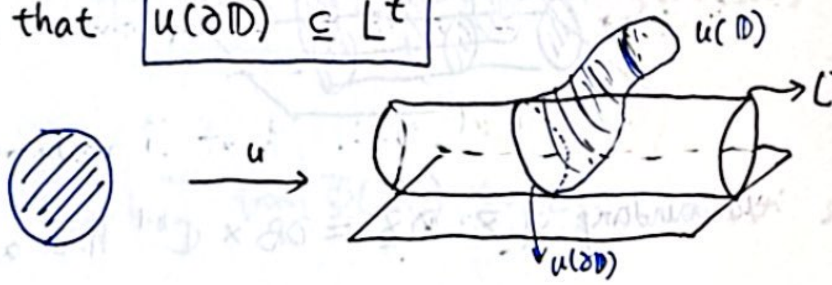
<u>Def</u>: The Moduli space $\mathcal{M}$ is the set of J-holomorphic disc

$$u: (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{R}^{2n}, \partial Z)$$

with the following properties.

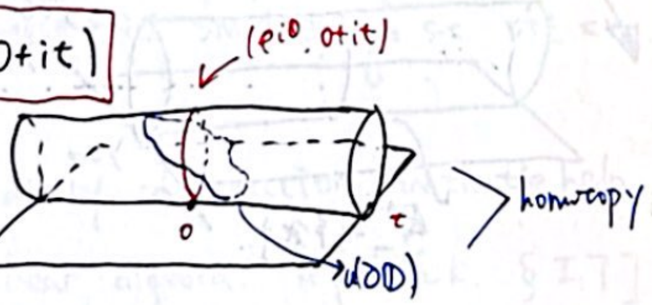
(M1) (Lagrangian boundary condition)

For every $u \in \mathcal{M}$ there is a $t \in \mathbb{R}^{n-1}$, called the level of u. such that $\boxed{u(\partial\mathbb{D}) \subseteq L^t}$.



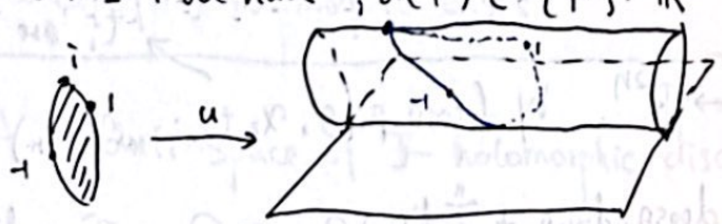
(M2) (Homotopy condition)

The boundary map $u|_{\partial\mathbb{D}}: \partial\mathbb{D} \to L^t$ is (smoothly) homotopic to $\boxed{e^{i\theta} \mapsto (e^{i\theta}, 0+it)}$



(M3) (three-point condition)

For $k = 0, 1, 2$, we have, $u(i^k) \in \{i^k\} \times \mathbb{R}^{n-1} \times \{y = t\}$



Rmk: ① (M1) is essential for the analytical set up.

② (M2) is will given the property of symplectic energy π.

③ (M3) is somehow like normalization. Since the automorphism of disc is determined by 3-points (Möbius transformation).

The central thm of this note is

> **Thm**: For a generic choice of the ω-compatible almost complex structure J, the mouduli space $\mathcal{M}$ of J-holomorphic discs is a smooth mfld of dim $2n-2$.
>
> The evaluation map $ev: \mathcal{M}\times\mathbb{D}\to\mathbb{R}^{2n}$. $(u,z)\mapsto u(z)$ is smooth, proper, and it takes value in $\overline{Z}$. Regarded ev as $\mathcal{M}\times\mathbb{D}\to\overline{Z}$. we have the mod 2 $\deg_2 ev = 1$.

FACT: This thm is really hard.

Corollary: Possibly after reparametrising with an element of $\mathrm{Aut}(\mathbb{D})$. there is a J-holomorphic disc $\hat{u}:(\mathbb{D},\partial\mathbb{D})\to(\overline{Z},\partial Z)$. $\hat{u}(0)=\Phi(0)$.

(This is not a direct corollary for now. after we talk more about the evaluation map, this may be clear)

We already "can" admit $\exists$ a J-holomorphic disc satifies (D1) & (D3). now we see that (D2) is also holds thanks to (M2)

<u>Prop</u>: Every $u \in \mathcal{M}$ has symplectic energy equal to π.

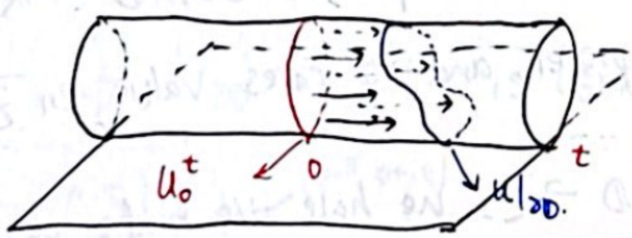
$$\int_{\mathbb{D}} u^*\omega = \pi.$$

<u>Proof</u>: Let $u \in \mathcal{M}$ be a J-holomorphic disc of level $t \in \mathbb{R}^{n-1}$ in (M1). Set $u_0^t(z) := (z, 0+it) = (z, it) \in L^t$, $z \in \mathbb{D}$

Thanks to homotopy condition (M2), we can find a homotopy

$$H: [0,1] \times \partial\mathbb{D} \longrightarrow L^t$$

from $H(0,\cdot) = u_0^t|_{\partial\mathbb{D}}$ to $H(1,\cdot) = u|_{\partial\mathbb{D}}$.



Note that $\omega = \sum_{i=1}^n dx^i \wedge dy^i = d\left(\sum_{i=1}^n x^i \wedge dy^i\right) =: d\lambda$. hence

$$\int_{\mathbb{D}} u^*\omega = \int_{\mathbb{D}} d(u^*\lambda) = \int_{\partial\mathbb{D}} u^*\lambda = \underbrace{\int_{[0,1]\times\partial\mathbb{D}} H^*\omega}_{\overset{\shortparallel}{0}} + \int_{\partial\mathbb{D}} (u_0^t)^*\lambda$$

since ω is Lagrangian on L^t

$$\Rightarrow \int_{\mathbb{D}} u^*\omega = \int_{\partial\mathbb{D}} (u_0^t)^*\lambda = \int_{\mathbb{D}} (u_0^t)^*\omega = \int_{\mathbb{D}} dx \wedge dy = \pi.$$

$\square$