Hyperbolic Geometry - 1

Main Ref: << An Introduction to Geometric Topology>> Martelli

· Roughly speaking, there are "three" kind of geometries in Riemannian geometry: S^n. $\mathbb{R}^n$. $\mathbb{H}^n$, responding to the unique simply connected, complete, Riemannian mfld with constant sectional curvature 1, 0, -1 (positive, flat, negative)

In this seminal, I will introduce several Hyperbolic model. I will list them as below. They have different advantages and drawbacks. we will use one of them as long as it is convienient.

	$\boxed{\text{Model name}}$		$\boxed{\text{advantage}}$
1:	Hyperboloid	⟶	easy to calculate
	↓ (projection)		
2:	projection poincaré disk	⟶	easy to see
	↓ (inversion)		
3:	Half plane model	⟶	has a nice ∞
4:	Klein model	⟶	geodesics are "lines" (easy to see)

And we will follow the path above to study them. In each model, we will care about

① How does they look like?

② What is the subspace? (geodesically submfld)

③ the metric tensor calculation.

④ Some geometric properties.

§1 Hyperboloid model ($\mathbb{I}^n$)

Recall how we get S^n? From Euclidean inner product $\langle x,y\rangle_E := \sum_{i=1}^{n} x_i y_i = 1$. So we can generalize this thing. Firstly, we give the defn of "scalar product".

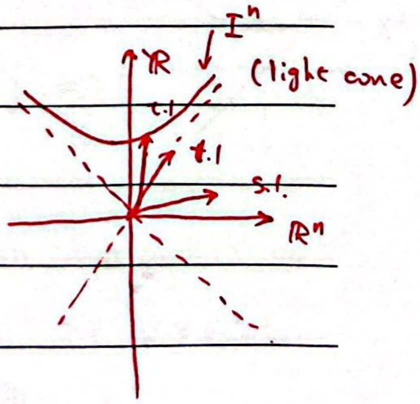
$\boxed{\text{Def 1}}$ (Scalar product)

Let V be an vector space. $\langle , \rangle$ is a symmetric bilinear form and nondegenerate. we call it is a scalar product with singature (p,q). ($p+q=\dim V$) if it can be represented by $\begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$ for some special basis.

In $\mathbb{R}^{n+1}$. $\langle x \cdot y\rangle_L := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$ is called the Lorentzian scalar product. its signature is $(n,1)$. $\langle x \cdot y\rangle_E := \sum_{i=1}^{n+1} x_i y_i$ is the traditional inner product. its ... is $(n+1,0)$.

$\boxed{\text{Def 2}}$ (Hyperbolid $\mathbb{I}^n$)

$$\mathbb{I}^n := \{ x \in \mathbb{R}^{n+1} \mid \langle x.x\rangle_L = -1,\ \underline{\underline{x_{n+1} > 0}} \}$$



- time-like vector, if $\langle x.x\rangle_L < 0$
- light-like vector, if $\langle x.x\rangle_L = 0$
- space-like vector, if $\langle x.x\rangle_L > 0$.

$\boxed{\text{Naive Question}}$: Does I^n is a mfld? Riemmainn Mfld?

$\boxed{\text{Prop3}}$: For any scalar product $\langle , \rangle$ on $\mathbb{R}^{n+1}$, The function $f: \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x) = \langle x, x \rangle$ is everywhere smooth and has differential, $\forall\, y \in T_x\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$

$$f_{*x}(y) = 2\langle x, y\rangle.$$

Pf: consider $r(t) = x + ty$. then

$$f_{*x}(y) = f_{*x}(\dot{r}(0)) = \frac{d}{dt}\Big|_{t=0} f \circ r(t)$$

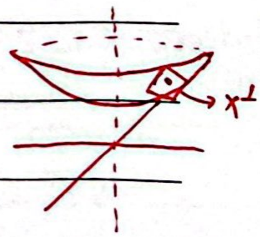
$$= \frac{d}{dt}\Big|_{t=0} \langle x+ty, x+ty \rangle = 2\langle x, y\rangle. \qquad \square$$

$\boxed{\text{Corollary4}}$ I^n is a Riemannian Mfld.

Pf: ① Since $\langle , \rangle_L$ is nondegenerate $\Rightarrow f_*$ is surjective $\Rightarrow$ -1 is the regular value of f. $\Rightarrow I^n$ is a mfld.

② To check I^n is a Riemannian mfld

$\boxed{\text{Step 1}}$: what is the tangent space?



$$\underbrace{T_x I^n = \ker f_{*x}}_{} = \{y \mid \langle x, y\rangle_L = 0\} := x^\perp$$

That's why we first introduce this model. because the tangent space has a nice straightforward expression!

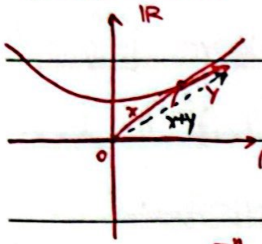
(level set)

(this is really similar to S^n!!)

$\boxed{\text{Step2}}$: what is the metric tensor $\Leftrightarrow$ Find an inner product on $x^\perp$.

$\boxed{\text{Claim:}}$ $\langle , \rangle_L \big|_{x^\perp}$ gives an inner product. i.e. $\forall\, y \in x^\perp$. $\langle y, y\rangle_L \geq 0$. and $= 0$ iff $y = 0$.

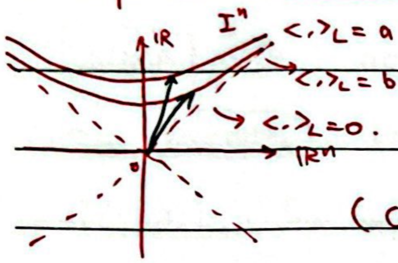
<u>pf of claim</u>: (Note that $\langle x, y\rangle_L = 0$ although they are not look like "or")



Goal: $\langle y, y\rangle_L \geq 0$. consider $\langle x+y, x+y\rangle_L$.

$\Leftrightarrow$ pf: $\langle x+y, x+y\rangle_L \geq \langle x, x\rangle_L$.

$\boxed{\text{observation}}$: $x+y$ lies below the I^n!!



$\Rightarrow \langle x+y, x+y\rangle_L \geq \langle x, x\rangle_L$.

Then we finish!!

(Check the verison of sphere!)

$\square$

「Exercise: For different scalar product $\langle , \rangle$, $\langle x, x\rangle = c$ diffeomorphic iff signature same」

Now we study the $\boxed{\text{Isometries}}$ of I^n.

$\boxed{\text{Defn 5}}$: $f: I^n \to I^n$ is called an Isometry if

① it is a diffeomorphism.

② $\forall x \in I^n$, $u, v \in x^{\perp} = T_x I^n$, $\langle f_{*x}(u), f_{*x}(v)\rangle_L = \langle u, v\rangle_L$

a Riemannian Exercise.

$\Longleftrightarrow \forall p, q \in I^n$, $\langle f(p), f(q)\rangle_{I^n} = \langle p, q\rangle_L$

$\Longleftrightarrow \langle AX, AY\rangle_L = \langle X, Y\rangle_L$

$\boxed{\text{Thm 6}}$: $\mathrm{Isom}(I^n) = O^+(n,1) = \{A \mid A^T \begin{pmatrix} I_n & 0 \\ 0 & -1\end{pmatrix} A = \begin{pmatrix} I_n & 0 \\ 0 & -1\end{pmatrix}$, A presene the upper half$\}$.

这两个条件说的一回事. 等价完全由一阶信息决定 → 切映射给出.

($\Leftarrow$) trivial $O^+(n,1) \subset \mathrm{Isom}(I^n)$

($\Rightarrow$) We will use a fact from Riemannian mfld: 「If M, N are connected Riemannian mfld. f, g are two isometries. $f(p) = g(p)$, $f_{*p} = g_{*p}$, then $f = g$.」 Thus <u>we only need</u> to For any $f \in \mathrm{Isom}(I^n)$, Find a matrix $A \in O^+(n,1)$ and some $p \in I^n$

① $f(p) = A(p)$ ② $f_{*p} = A$. $\rightsquigarrow$ then $f = A$

Now we have $f \in Isom(I^n)$ is arbitrary. How to find A?

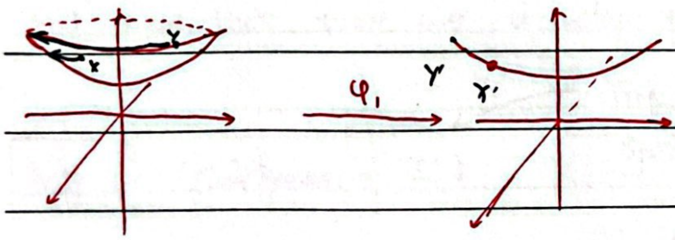
$\boxed{\text{Step 1}}$ specialize f. we also need to find some p. we also need to find a special p. ⟶ chose $x_0 = (0,\cdots,0,1)$. and $x_1 := f(x_0)$.

~~• $\exists p_1, f(p_1) = x_0$ (f is diffeomorphism.~~

$x_1 \longrightarrow x_0$.

$\boxed{\text{LEMMA}}$: $O^+(n,1)$ acts transitively on I^n ⟷ thus we can transform

$\boxed{\text{Key observation}}$: we can rotate first n- coordinate. i.e. $\begin{pmatrix} A & \\ & 1 \end{pmatrix}$, $A \in O(n)$ $\in O^+(n,1)$. For any $x, y \in I^n$. $\exists \varphi_1 \in O^+(n,1)$. such that think sphere



$x' = (x_1, 0\cdots 0, x_{n+1})$. $y' = (y_1, 0\cdots 0, y_{n+1})$

then we have $x_1^2 - x_{n+1}^2 = -1$

$y_1^2 - y_{n+1}^2 = -1$

旋转过这两点的大圆 =.

Let $x_1 = \sinh t_1$, $x_{n+1} = \cosh t_1$, $y_1 = \sinh t_2$, $y_2 = \cosh t_2$, we have another rotation:

$$\begin{pmatrix} \cosh(t_2 - t_1) & 0 & \sinh(t_2 - t_1) \\ 0 & I_{n-2} & 0 \\ \sinh(t_2 - t_1) & 0 & \cosh(t_2 - t_1) \end{pmatrix} := \varphi_2.$$ 双曲旋转.

sends x' to y' (Direct Calculation) □

$\boxed{\text{启发:}}$ 将双曲线视作大圆!! (类比 S^n)

Now consider $\psi \in O^+(n,1)$. and $\psi(x_1) = x_0$. so we have $\psi \circ f \in Isom(I^n)$. $\psi \circ f(x_0) = x_0$.

$\to T_{x_0} I^n = \{(x_1, \cdots x_n, 0)\}$.

$\boxed{\text{Step 2}}$: suppose $(\psi \circ f)_{*x_0} = B$. then $B \in O(n)$. (direct calculation)

$\Longrightarrow$ chose $C = \begin{pmatrix} B & \\ & 1 \end{pmatrix} \in O^+(n,1)$. then

① $(\psi \circ f)_{*x_0} = C_{*x_0} = C$. ② $\psi \circ f(x_0) = x_0 = C(x_0) = x_0$.

$\Longrightarrow \psi \circ f = C \Longrightarrow f = \psi^{-1} C \in O^+(n,1)$! □

Rmk: $O(n) \subseteq O^+(n,1)$. We have not a explicit formula of $O^+(n,1)$. So we still not know the explicit Isometry group. Actually they are quite complicated. We will only talk about the lower dimension isometry group.

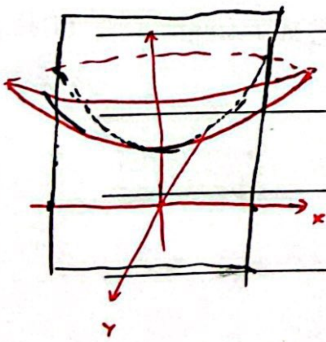
Now we Introduce the "subspaces" of I^n. More precisely we will define some submflds of I^n directly. And then we will show that they are actually geodesically.

You can view it as some special submflds if you have not study RG.

$\boxed{\text{Defn 7}}$ (subspaces of I^n)

A k-dimensional subspace of I^n is the intersection of a $\boxed{(k+1)}$-dimensional vector subspace of $\mathbb{R}^{n+1}$ with I^n.

contain origin!!!! We will only say "affine" if it does not contain

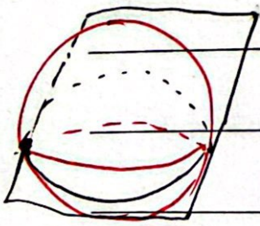


$\boxed{\text{Rmk:}}$ For $k+1$ - vector subspaces $W \subseteq \mathbb{R}^{n+1}$ Following are Eq..

① $W \cap I^n \neq \emptyset$

② W contains at least a time-like vector

③ the signature of $\langle,\rangle|_W$ is $(k,1)$.



Let $W \cap I^n = S$. then $\forall x \in S$, $\underrightarrow{T_x S} \subseteq W$. k.dim. $x \in S$

We can choose $\{x, T_x S\}$ as a basis of W.

From the rmk. we have an important corollary:

Corollary 8 A k-subspace of I^n is itself isometric to I^k.

The non-empty intersection of two subspaces is always a subspace.

An isometry of $\mathbb{R}^n$. I^n sends k-subspace to k-subspace

Defn 9 (lines) A 1-subspace is a line. i.e. a 2-vector subspaces intersection with I^n.

(the line in $\mathbb{R}^n$ are the line. the line is the S^n are the big circle. the line in I^n are the hyperbolic lines).

为了照顾未学RGeo同学采用如此的定义

Now we talk about the geodesics. We can a $\gamma(t) \subseteq M^n \subseteq \mathbb{R}^{n+1}$ is a geodesic if ① $\gamma(t) \subseteq M^n$ ② $\langle \dot\gamma(t), \dot\gamma(t) \rangle = 1$. ③ $\ddot\gamma(t)$ projects to the tangent space are zero. 对于 $\mathbb{R}^n$ 中直线. 其加速度在切向无作用.

$\langle V, V \rangle_E = 1$, $\langle V, V \rangle_L = 1$

Thm 10: Let $p \in M$. be a point and $V \in T_pM$ a "unit" vector. The geodesic γ exiting from p with velocity V is

- $\gamma(t) = p + tV$ if $M = \mathbb{R}^n$
- $\gamma(t) = \cos t \cdot p + \sin t \cdot V$ if $M = S^n$
- $\gamma(t) = \cosh t \cdot p + \sinh t \cdot V$ if $M = I^n$.

Pf: We take $\mathbb{I}^n$ as an example. What we need to show?

① $\gamma(t) \subseteq \mathbb{I}^n$. $\Leftrightarrow$ $\langle \gamma(t), \gamma(t)\rangle_L = -1$

$\langle \cosh t \cdot p + \sinh t \cdot v, \cosh t \cdot p + \sinh t \cdot v\rangle_L = (\cosh t)^2 \cdot (-1) + (\sinh t)^2 \cdot (1) = -1$

② $\dot\gamma(t) = \sinh t \cdot p + \cosh t \cdot v$ is unit ~~speed~~ vector.

$\langle \dot\gamma(t), \dot\gamma(t)\rangle = -\sinh^2 t + \cosh^2 t = 1$.

③ $\ddot\gamma(t) = \cosh t \cdot p + \sinh t \cdot v. = \gamma(t)$. $\gamma(t) \perp T_{\gamma(t)}\mathbb{I}^n \Rightarrow$ ✓.

Then $\gamma(t)$ is the geodesic.

$\boxed{\text{Thm 11}}$ all geodesics $\Leftrightarrow$ lines.

- From linearity $\Rightarrow$ lines are geodesics $\cosh t \cdot p + \sinh t \cdot v \subseteq W$.
- Suppose $W = \mathrm{span}\{p, v\}$. then we need to check

$$W \cap \mathbb{I}^n = \{\cosh t \cdot p + \sinh t \cdot v\}.$$

"$\supseteq$" is trivial.

"$\subseteq$" we need to paramterize the $W \cap \mathbb{I}^n$. $\forall q \in W \cap \mathbb{I}^n$, we can use $O^+(n,1)$ tends $\ell = W \cap \mathbb{I}^n \longrightarrow x_1^2 - x_{n+1}^2 = -1$.

$\rightsquigarrow \sinh t, \cosh t$ —— we can easily ... □

$\boxed{\text{Prop 12}}$ Let $p, q \in M$. we have

- $M = S^n$. $\cos(d(p,q)) = \langle p, q\rangle_E$
- $M = \mathbb{I}^n$. $\cos(d(p,q)) = -\langle p, q\rangle_L$.