Chapter 9 Topology of three-manifolds

§ 9.1 Algebraic Topology.

Basic FACT: If M is a closed orientable 3-mfld.

$\Rightarrow H_*(M)$ $(H^*(M))$ is determined by $\pi_1(M)$.

- $H_1(M) \cong \pi_1(M)^{ab}$. $H_2(M) \cong H^1(M) \cong H_1^{Free}(M)$.

  $H_0(M) \cong H_3(M) \cong \mathbb{Z}$.

Prop 1: • If M is a closed 3-mfld $\Rightarrow \chi(M) = 0$

• If M is a compact 3-mfld with boundary

$$\Rightarrow \chi(M) = \frac{1}{2}\chi(\partial M)$$

Prop 2: Consider the boundary map of long exact sequence of $(M, \partial M)$, i.e. $\partial: H_2(M, \partial M) \to H_1(\partial M)$.

We have $\mathrm{Im}\,\partial$ is a Lagrangian subgrp of $H_1(\partial M)$.

Rmk: Since ∂M is an oriented closed surface $\Rightarrow H_1(\partial M) \cong \mathbb{Z}^{2g(\partial M)}$, with a natural symplectic structure — intersection form. And For a symplectic vector space (V^{2n}, ω), $W \subseteq V$ is called Lagrangian if $\dim W = n$. and $\omega|_W \equiv 0$.

Before we give the proof of Prop 2. we will firstly give a geometric interpretation of $H_{n-1}(M, \partial M)$ generally.

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- By Poincaré-Lefschetz Duality:

$$H_{n-1}(M, \partial M) \cong H^1(M) \cong [M, S^1]$$

Hence $\forall\, \alpha \in H^1(M)$, $\exists\, f \in [M, S^1]$ s.t. $f^*[S^1] = \alpha$. Since M and S^1 are smooth, then by a well-know result, $\exists$ smooth map homotopic to f. Then by Sard's lemma, $\exists\, p \in S^1$ is the regular value

$\Rightarrow \Sigma = f^{-1}(p)$ is a regular hypersurface in M. Now we have i: $\Sigma \hookrightarrow M$, and $i_*[\Sigma] \in H_{n-1}(M, \partial M)$, and actually,

$$[\Sigma] = [f^{-1}(p)] \Rightarrow PD.[\Sigma] = PD.[f^{-1}(p)] = f^*(PD.[p])$$

$\Rightarrow PD.[\Sigma] = f^*([S^1]) = \alpha \Rightarrow \Sigma$ can represent $PD.\alpha$ ##

Rmk: Actually, $\partial[\Sigma] = [\partial\Sigma]$, this can be seen by diagram chasing.

Proof of Prop 2: we have the long exact seq.

$$\cdots \to H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i_*} H_1(M) \to \cdots$$

and two pairings: $\left\{ \begin{array}{l} \omega: H_1(\partial M) \times H_1(\partial M) \to \mathbb{Z} \\ \eta: H_2(M, \partial M) \times H_1(M) \to \mathbb{Z} \end{array} \right.$ → Poincaré-Lefschetz duality

all denotes by intersection number

FACT: $\forall\, a \in H_2(M, \partial M)$, $b \in H_1(\partial M)$, we have

$$\omega(\partial a, b) = \eta(a, i_* b).$$

This can be seen directly from the geometric meaning, since b is a curve lie on the boundary, a is a regular embedded surface.

$\#(a \cap b)$ only happen on the boundary. and $a \cap \partial M = \partial a$. then we know the proof.

From the equality, Now we can prove that $\mathrm{Im}\,\partial$ is Lagrangian for ω.

by the exactness.

- $\omega(\partial a, \partial b) = \eta(a, i_*\partial b) = \eta(a, 0) = 0$ Half lives-half dies thm.

- Next goal is to show that $\dim \mathrm{Im}\,\partial = \frac{1}{2} \dim H_1(\partial M)$

we already know that $\mathrm{Im}\,\partial$ is isotropic $\Rightarrow \dim \mathrm{Im}\,\partial \le \frac{1}{2} \dim H_1(\partial M)$.

If $\dim \mathrm{Im}\,\partial < \frac{1}{2} \dim H_1(\partial M)$.

$\therefore$ By $\ker i_* \cong \mathrm{Im}\,\partial \Rightarrow \dim \ker i_* < \frac{1}{2} \dim H_1(\partial M)$;

$\therefore$ By $H_1(\partial M)/\ker i_* \cong \mathrm{Im}\, i_* \Rightarrow \dim \mathrm{Im}\, i_* > \frac{1}{2} \dim H_1(\partial M)$;

$\therefore$ By $H_2(M, \partial M)/\ker \partial \cong \mathrm{Im}\,\partial \Rightarrow \dim \ker \partial > \dim H_2(M, \partial M) - \frac{1}{2} \dim H_1(\partial M)$
$= b_1(M) - \frac{1}{2} b_1(\partial M)$;

FACT: If $a \in \ker \partial$, $b \in \mathrm{Im}\, i_*$ $\Rightarrow \eta(a, b) = \eta(a, i_* \tilde{b}) = 0$.

$\Rightarrow \ker \partial$ and $\mathrm{Im}\, i_*$ are η-orthogonal

$\Rightarrow \ker \partial \oplus \mathrm{Im}\, i_* \subseteq H_1(M) \cong H_2(M, \partial M)$

$\Rightarrow b_1(M) < \dim \ker \partial + \dim \mathrm{Im}\, i_* \le \dim H_1(M) = b_1(M)$.

which is a contradiction!

Finally we prove that $\mathrm{Im}(H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M))$ is a Lagrangian subspace of $H_1(\partial M)$. $\square$

Corollary 3: If M is an oriented connected 3-mfld

$$\Rightarrow \quad b_1(M) \geq \frac{1}{2} b_1(\partial M)$$

Proof: Recall in the

$$\cdots \to H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i_*} H_1(M) \to \cdots$$

we have $\dim \operatorname{Im} \partial = \frac{1}{2} b_1(\partial M)$. and

$$H_2(M, \partial M) / \ker \partial \cong \operatorname{Im} \partial \Rightarrow 0 \leq \dim \ker \partial = b_1(M) - \frac{1}{2} b_1(\partial M)$$

which is as desired. $\square$

Corollary 4: The boundary of simply connected 3-mfld consists of spheres.

- The orientable M^3 may contain non-orientable surface

Ex 5:

the Möbius band in the solid torus.

Ex 6: $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$.

If $S \hookrightarrow M^3$ is orientable, then by the tubular nbhd thm, its tubular nbhd $\cong$ its normal bundle. i.e. a trivial bundle $\Rightarrow \cong S \times I$.

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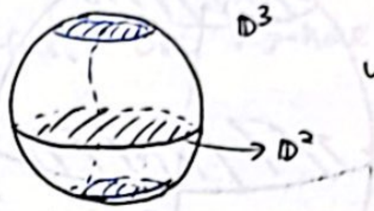
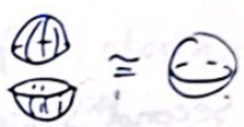
But if S is non-orientable, then the tubular nbhd of S is an orientable line bundle over S (since by tubular nbhd thm, the normal line bundle $\cong$ an open domain in M^3, hence orientable).

And for non-orientable surface, this line bundle is unique, we denote it by $S \tilde{\times} I$. with boundary is the 2-fold orientable covering of S. (Recall the line bundle of $X \longleftrightarrow$ 2-fold cover of X)

EX7: The tubular nbhd of $\mathbb{RP}^2$ in $\mathbb{RP}^3$

view $\mathbb{RP}^2 \hookrightarrow \mathbb{RP}^3$ as  with anti-attaching

Firstly, the line bundle of D^2 is as above. then we can see that after quotient, the boundry of $\mathbb{RP}^2 \tilde{\times} I$ is the  S^2.

Prop 8: Let M^3 be orientable, Then each non orientable surface S determines a non-trivial class $[S] \in H_2(M, \partial M; \mathbb{Z}_2)$.

Then M^3 contains at most $\dim H_2(M, \partial M; \mathbb{Z}_2) = \dim H_1(M)$ disjoint non-orientable surfaces.

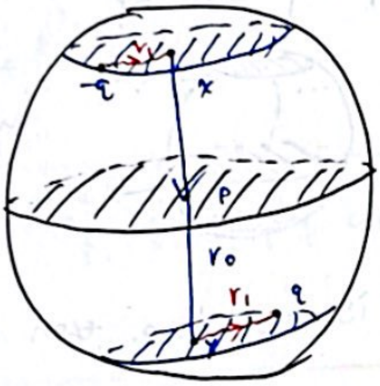
PROOF: For the first statement, we argue it by contradiction.

If $[S]$ is trivial, then $\forall\, a \in H_1(M; \mathbb{Z}_2)$, $\eta([S], a) = 0$.

i.e. a intersects S even times.

However, consider the tubular nbhd of S in M^3. i.e. $S \tilde{\times} I$. we can find a closed curve a in $S \tilde{\times} I$ intersects S only once. It is easy. $\forall p \in S$. consider $x = (p, 1)$. and $y = (p, -1)$. the endpoints of the line perp. and $r_0 = \overline{xy}$ the line. Since $\partial(S \tilde{\times} I)$ is connected. we can find another curve on $\partial(S \tilde{\times} I)$ connecting y to x. denote it as r_1. then $r_0 * r_1$ is as desired.

「A more accurate construction of $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$:



」

The second statement is trivial to consider $S = \bigsqcup_{i=1}^{n} S_i$ □

Corollary 9: A simply-connected 3-mfld does not contain any closed non-orientable surface!

§ 9.2 Prime Decomposition

In this section, we will mainly talk about the "easiest" pieces of 3-Mfld. $\rightsquigarrow$ irreducible & prime.

I: Irreducible mfld & Alexander Thm.

A general strategy we use in 3-mfld is to firstly remove a handlebody then glue it back. Actually, how to glue back a solid torus $\overset{\text{almost}}{\Longleftrightarrow}$ a 3 mfld. But we have a easy but important fact:

FACT 1: The mfld M obtained by capping off a spherical boundary component of N does not on the diffeo chosen. i.e. If φ, ψ are two diffeo of N ($\cong S^2$), then $M \cup_\varphi D^3 \overset{\text{diff}}{\cong} M \cup_\psi D^3$.

Rmk: easy fact, if φ and ψ are isotopic, then trivially $M \cup_\varphi \tilde{M} \overset{\text{diff}}{\cong} M \cup_\psi \tilde{M}$.

step 1: Brouwer $\rightsquigarrow$ a fix point
step 2: Alexander trick for D^2 $\Rightarrow$ (id & $-$id up to isotopic)

Since the mapping class grp of S^2 is trivial. It suffices to prove $M \cup_{id} D^3 \overset{\text{diff}}{\cong} M \cup_{-id} D^3$, which is trivial. □

> Rmk: From now on, in dim 3, we can freely remove and add Balls without affect the topology of the mfld.

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With this fact in mind, we can know that the connected sum below is well-defined.

Def 2: The connected sum $M_1 \# M_2 =: M$ of two oriented connected 3-mflds M_1, M_2 is constructed by removing the interiors of two closed balls from M_1 and M_2, and then gluing the two resulting spheres via any orientation reversing diffeo. $\Rightarrow$ to make sure M has a canonical orientation.

Rmk: Since Mapping class grp of S^2 is trivial $\Rightarrow$ all reversing diffeo of S^2 are isotopic $\Rightarrow$ the defn above is a good defn.

Exercise: An orientable mfld is called mirrorable if it admits an orientation reversing self diffeo (e.g. S^2 ✓, $\mathbb{CP}^2$ ✗).

Now If M_1 is mirrorable, then $M_1 \# M_2 \overset{\text{diff}}{\cong} \overline{M_1} \# M_2$.

Proof: suppose $f: M_1 \to \overline{M_1}$ is a diffeo. then $f: M_1 - B \to \overline{M_1} - f(B)$ is still a homeo. then

$$
\begin{aligned}
\overline{M_1} \# M_2 &\cong (\overline{M_1} - f(B)) \cup (M_2 - B) \\
&\quad\Big\downarrow f^{-1} \cup \mathrm{id} \\
&\cong (M_1 - B) \cup (M_2 - B) \cong M_1 \# M_2
\end{aligned}
$$

$\square$

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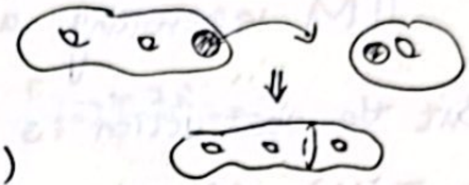
Def 3: The ∂-connected sum $M := M_1 \#_\partial M_2$ of two oriented 3-mflds with boundary is constructed by gluing two discs $D_1 \subset \partial M_1$ and $D_2 \subset \partial M_2$ via an orientation reversing diff.

FACT: $\cdot\ M \# S^3 = M$ 「Assume ∂M_1 and ∂M_2 are connected.

$\cdot\ M \#_\partial B = M$



$\cdot\ \partial(M_1 \#_\partial M_2) = (\partial M_1) \# (\partial M_2)$ 」

「If M_1, M_2 are both closed, oriented」

$\cdot\ \pi_1(M_1 \# M_2) \cong \pi_1(M_1) * \pi_1(M_2)$ $(\dim M_i \geq 3)$

$\cdot\ H_q(M_1 \# M_2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}. & q = 0. \dim M_i \\ H_q(M_1) \oplus H_q(M_2). & \text{otherwise} \end{cases}$

Exercise: Construct a 4-dim closed oriented connected mfld with the same homology grp as T^4, but not homeo.

Hint: $X = (\#_4 S^1 \times S^3) \# (\#_3 S^2 \times S^2)$ $\square$

Now we can start the topic of irreducible 3-mfld.

Def 4: The mfld M is a connected, oriented 3-mfld, it is irreducible if every embedded sphere $S \subseteq M^\circ$ bounds a Ball.

Rmk: "Easy" to see, M^3 is irreducible $\Rightarrow \pi_2(M) = 0$. 「this still needs Sphere thm. 」

Hence $S^2 \times S^1$ is not irreducible. 「this is only a quick observation. we will prove later」

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Rmk': I want to ask the inverse of the rmk above: If $\pi_2 M = 0$ $\overset{?}{\Longrightarrow}$ M^3 is irreducible. What I know is the following

Sphere Thm (a highly nontrivial result. [Hat-Thm 3.8])

> If M^3 cco, $\pi_2 M^3 \neq 0$, $\exists$ embedded sphere S^2 in M representing a nontrivial element in $\pi_2(M)$

But the obstruction is An embedded sphere $S^2 \subset M$ is zero in $\pi_2(M)$ iff it bounds a compact contractible submfld of M, but it may not be $\mathbb{D}^3$ (I'm not sure).

$\rightarrow$ capping it off. then get a simply connected 3-mfld $\overset{\text{by Poincaré}}{\underset{\text{conj.}}{\rightsquigarrow}} \cong S^3 \Rightarrow \mathbb{D}^3$!!

「I naively hope " M irreducible $\overset{?}{\Longleftrightarrow}$ $\pi_2 M = 0$ " 」

Rmk: THIS is TRUE!

Now forget the big thm above, we will use basic tools to find some irreducible 3-mfld.

Easy to see, the easiest mfld is $\mathbb{R}^3$.

Thm (Alexander) $\mathbb{R}^3$ is irreducible. i.e. each embedded sphere bounds a ball.

「Rmk: In $\mathbb{R}^2$. it has a baby version $\Rightarrow$ Jordan curve thm. 」

IDEA: use Morse height function to 分划球面，对局部标准情形填充圆盘以及实心材料.

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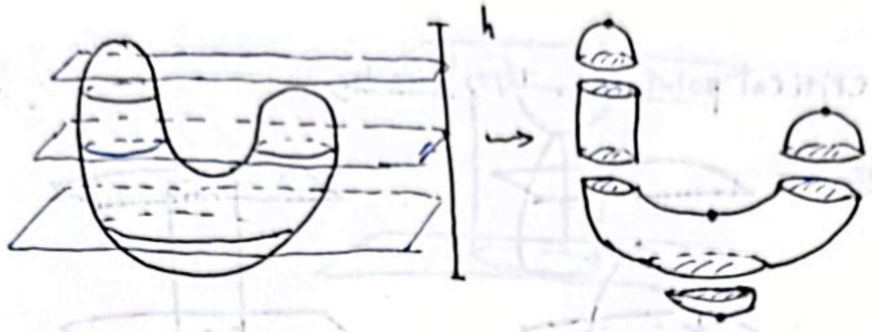


补回回去

⟶ 得到一个完整的球.

「Another possibilities (might ignore at first)

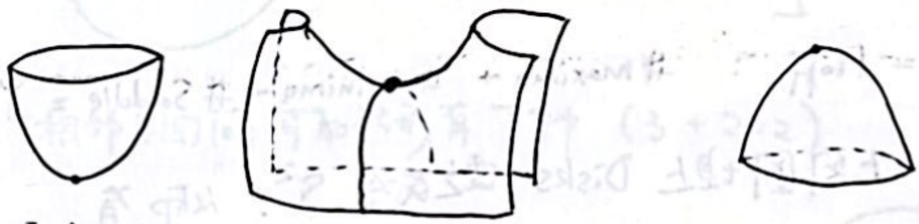
「向下戳了一下.」

此时填充就不单单是补洞,还需要挖去!!

Goal: ① Classification of possible local model.

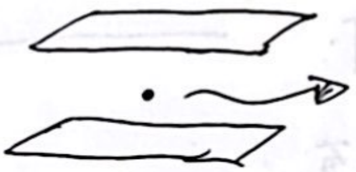
② Accurate surgery.

For ①: we use index (Morse) to classify locally:



index=0 index=1 index=2

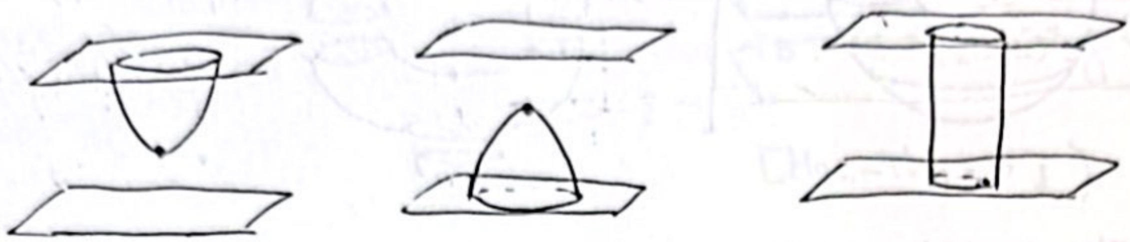
Now if



Critical points.

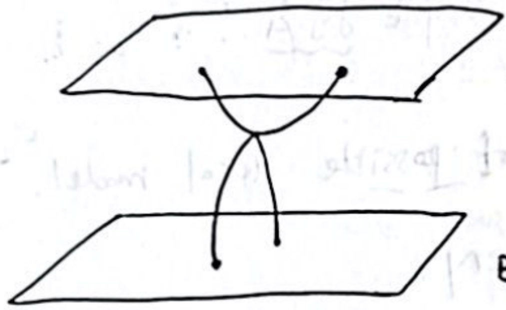
两平面之间可能夹着的曲面有哪些呢?

· 当 index=0.2 或无 critical point 时. 均为显然的



Rmk: last case from Morse theory. if no critical points. then they are homotopic equivalent. (gradient flow gives)

· 当 index=1 时. 情形较为复杂.



A上可能有2个或1个圆. B上可能有1个或2个圆.

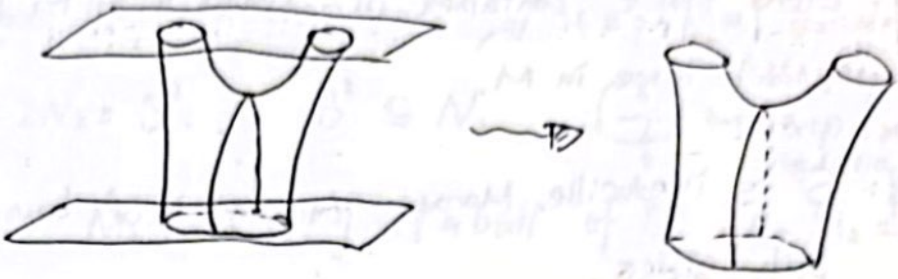
∵ 由 Poincaré - Hopf : $\#\text{maxima} + \#\text{minima} - \#\text{Saddle} = \chi(S)$

分别在A.B上圆圆贴上 Disks 使之成为 S^2. 从而有

$$\#A\text{中圆} + \#B\text{中圆} - 1 = 2$$

⇒ 只能 A 2个 B 1个 或者 A 1个 B 2个.

Case 甲: 若A 2个. B 1个 圆 从而有

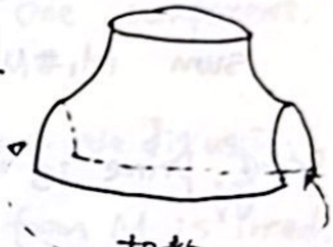


Case 2：若 A 1个，B 两个. 从局部上看

则需要上面两个半圆粘在一起.

修剪：

（也可看成是

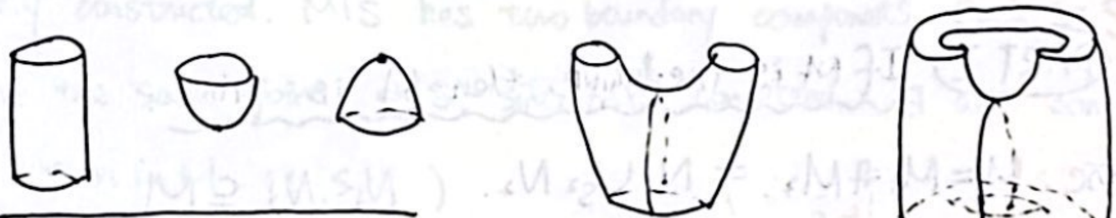


相粘）

末洞子

「可下戳的可能性」

Finally：相邻半圆间可能情形有7种 (3+2×2)



以及上下颠倒 （为保证 index=−1）

##

Corollary: Every sphere contained in a subball of M. bounds a ball in the subball. hence in M.

Corollary: S^3 is irreducible, Moreover, the sphere in S^3 bounds a ball on both sides.

II. Prime Mflds and more irreducible mflds

Def: • A connected sum $M_1 \# M_2$ is trivial if either M_1 or M_2 is a sphere.

• A connected, oriented 3-mfld M is prime if every connected sum $M_1 \# M_2$ is trivial.

Being prime is "almost" equivalent to irreducible, with only one single exception.

Prop: Every oriented 3-Mfld $M \neq S^2 \times S^1$ is prime iff irreducible. And $S^2 \times S^1$ is prime but reducible.

(Recall a quick cretion to distinguish reducible is $\pi_2 \neq 0$. and $\pi_2(S^2 \times S^1) \neq 0$).

Proof: PART I If M is irreducible, then M is prime.:

suppose $M = M_1 \# M_2 . = N_1 \cup_{S^2} N_2$. $N_1, N_2 \subseteq M$

Since S^2 bounds a ball. WLOG. its bounds a ball in N_2 side

Lem: $\partial M = \partial N$, $M \subseteq N \Rightarrow M = N$

pf: if $\exists p \in N \setminus M$, consider q.

$d(p,q) = d(p, \partial M)$

but $\exists B_\varepsilon(p) \cap \partial M = \phi$.

contradiction!

$\Rightarrow$ we have a ball B^3 such that

$$\partial B^3 = \partial N_2 = S^2. \qquad B^3 \subseteq N_2.$$

$\Rightarrow B^3 = N_2 \quad \Rightarrow \quad M_2$ is capping off a ball of B^3. i.e. $M_2 = N_2 \cup_{S^2} B^3$

$\cong B^3 \cup_{S^2} B^3 \cong S^3 \quad \Rightarrow$ the connected sum is trivial.

$\Rightarrow$ M is prime.

PART II If M is prime, but not irreducible $\Rightarrow M \cong S^2 \times S^1$

Claim: Consider the non-bound ball sphere S, then S must be non-separating. i.e. $M \setminus S$ has only one component.

pf: If $M \setminus S$ has two components, N_1 and N_2. from the discussion above we know that N_1, N_2 are all not balls from M is not irred.

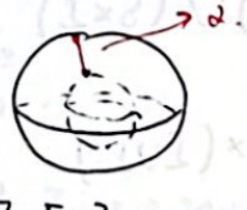
but this means that $M_1 = N_1 \cup_{S^2} B^3$, $M_2 = N_2 \cup_{S^2} B^3$ have

$M = M_1 \# M_2$ and M_1, M_2 are all not S^3. $\Rightarrow M$ is not prime

Hence, since S is non separating. $\exists$ a closed curve α in M.

s.t. α intersects S transversly in one point. (these can be directly constructed. $M \setminus S$ has two boundary components. each $\cong S$. choose the same point on S. Since $M \setminus S$ connected. $\exists$ arc connecting them. then in M, it is α)

$S^2 \times S^1$ cut S^2 $\sim$



$S^2 \times (0,1)$

May be these can seen by $[S] \cdot [\alpha] = 1$ use some algebraic topology, but I can not figure it out.

Now consider two tubular nbhds of S and a. $\mathcal{N}(S)$. $\mathcal{N}(a)$

$\mathcal{N}(S) \cong S^2 \times I$. $\mathcal{N}(a) \cong \mathbb{D}^2 \times S^1$. since $\mathcal{N}(S) \cap \mathcal{N}(a)$

is a solid cyclinder (one point $\rightsquigarrow$ Disk $\rightsquigarrow$ $D^2 \times I$)

$\Rightarrow \mathcal{N}(S) \cup \mathcal{N}(a)$ is attach another cyclinder away from

$S^2 \times I$. then the boundary is the connected sum of two

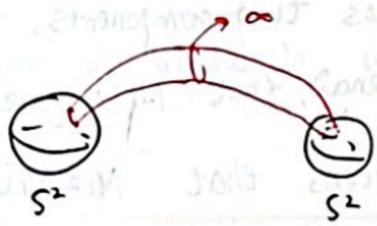
S^2. i.e. S^2 $\Rightarrow$ $\partial(\mathcal{N}(S) \cup \mathcal{N}(a)) =: S'$ is also a sphere.

「 picture :

$\to \mathcal{N}(S) \cup \mathcal{N}(a)$

「这里有个圆柱的入口」

$\partial(\mathcal{N}(S) \cup \mathcal{N}(a)) \cong$



Now S' is a seperting sphere in M. since M is prime,

then S' bounds a ball in the other side

$\Rightarrow M \cong \mathcal{N}(S) \cup \mathcal{N}(a) \cup B^3$

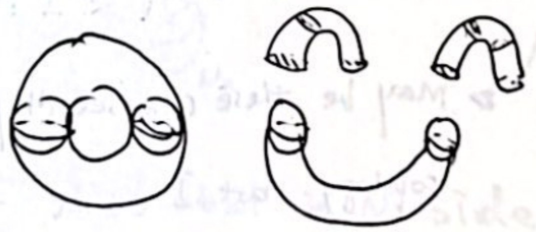
$\cong (S^2 \times I \sqcup \mathbb{D}^2 \times I) \cup_{S^2} (\mathbb{D}^2 \times I)$ 上下对照一下

粘贴的方法

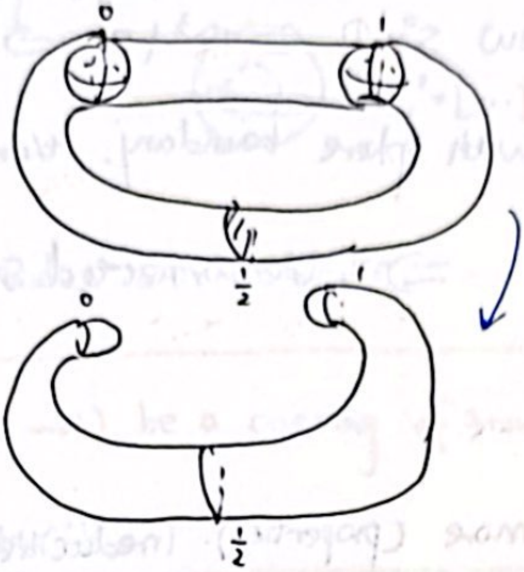
$\cong (S^2 \times I) \cup (S^2 \times I) \cong S^2 \times S^1$

$S^1 \times S^2 = (\mathbb{D} \cup \mathbb{D}') \times (I \cup I')$

$= \underbrace{(S^2 \times I) \cup \mathbb{D}^2 \times I' \cup \mathbb{D}^2 \times I'}$



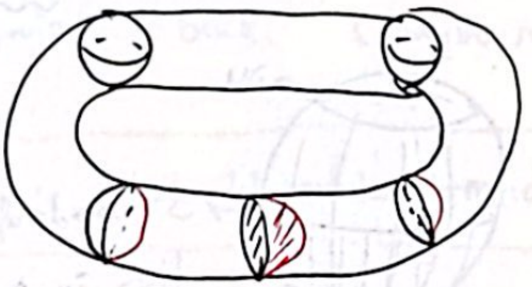
「中文解释: $(S^2\times I)\cup(\mathbb{D}^3\times I) = N(S)\cup N(0)$ 可化为



(*)

有毛扣圆也所

粘上一个 $\mathbb{B}^3$. 即相当于每个 $t\in[0,1]$. 再粘个 $\mathbb{D}^3$. 即在(*)想出上.



$\Rightarrow S^2\times S^1$. 」

PART III $S^2\times S^1$ is prime.

Consider $\forall$ separting sphere S. then $S^2\times S^1 \setminus S = U\cup V$.

$\partial U=\partial V = S$. From SVK $\Rightarrow \pi_1(U) * \pi_1(V) \cong \pi_1(S^2\times S^1)\cong \mathbb{Z}$

$\Rightarrow$ WLOG $\pi_1(U)=0$. consider $i: U \hookrightarrow S^2\times S^1$. and the universal cover of $S^2\times S^1$ is $S^2\times \mathbb{R}$.

$$\begin{tikzcd} & S^2\times\mathbb{R} \arrow[d, "\pi"] \\ U \arrow[ur, dashed, "I"] \arrow[r, hook, "i"] & S^2\times S^1 \end{tikzcd}$$

since $\pi_1(U)=0 \Rightarrow \exists$ lift I. and I is injective. since if $I(x)=I(y)$

$\Rightarrow i(x)=\pi\circ I(x) = \pi\circ I(y) = i(y) \Rightarrow x=y$ since i is embedding (inclusion)

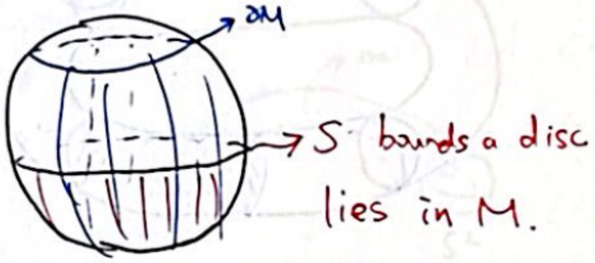
$\Rightarrow$ I is embedding $\Rightarrow U$ is a 3-submfld in $S^2\times\mathbb{R}$ with boundary $\partial U=S^2$. view $S^2\times\mathbb{R}\cong\mathbb{R}^3\setminus 0 \Rightarrow U$ is embedding in $\mathbb{R}^3\setminus 0$ hence $\mathbb{R}^3$. with sphere boundary. then by Alexander thm $\Rightarrow U\cong B^3$. $\Rightarrow$ the connected sum is trivial $\Rightarrow S^2\times S^1$ is prime.

$\square$

We know introduce more (properties) irreducible 3-mfld.

Prop: Every cmpt 3-dim submfld $M\subset S^3$ with connected boundary is irreducible.

Proof: Intutively:



Actually. Every sphere $S\subset M\subseteq S^3$ will bounds two balls at each side. Since ∂M is connected $\Rightarrow \partial M$ lies in one of the ball B, suppose the other ball is B', then $B'\subset M$. hence S bounds a ball in M.

「关于 $B'\subset M$. 仍然可用之前的方法证明. 若∃在内点 $p\in\mathring{B}'$ 但 $p\notin M$. 由 B' 闭.

不妨设 $d(p,M)=\max\limits_{q\in B'} d(q,M)$, 由M紧. 可知 $\exists x\in M$. $d(p,M)=d(p,x)$

而p为内点. 从而 $\exists\,\delta$. $B_\delta(p)\subseteq\mathring{B}'$. 由于已在 $\mathbb{R}^3$ 中. 从而可沿着 $\overrightarrow{xp}$ 方向延长 $\frac{\delta}{2}$ 到点 $\tilde{p}$. 则 $d(\tilde{p},x)=d(p,x)+\frac{\delta}{2}$. 与max性矛盾!

」

$\square$

Rmk: If boundary is not connected, then the conclusion is false.

 $S^2\times[0,1]$.

Corollary: Handlebodies are irreducible.

Prop: Let $p: M \to N$ be a covering of 3-mflds. If M is irreducible $\Longrightarrow$ N is irreducible.

Sketch: consider the sphere in N can be lifted in M. and bounds a ball then project it back. (maybe need chose the innermost one)

##

Corollary: Elliptic, Flat. Hyperbolic 3-mfld are irreducible.

corollary

~~Prop~~: If $g\geq 1$. $S_g\times[0,1]$ is irreducible.

Proof: the universal cover of $S_g\times[0,1]$ is $\mathbb{R}^2\times[0,1]$. has interior $\cong \mathbb{R}^3 \Rightarrow \mathbb{R}^2\times[0,1]$ is irreducible. $\Rightarrow S_g\times[0,1]$ is. □.

Exercise: $b\geq 1$. $S_{g,b}\times[0,1]$ is homeomorphic to the handlebody with $g=2g+b-1$.

Pf: We have handle decomposition. e.g.

$S_{2,1}\cong$  i.e. $S_{g,b}\times[0,1]\cong$ 

is a handlebody.

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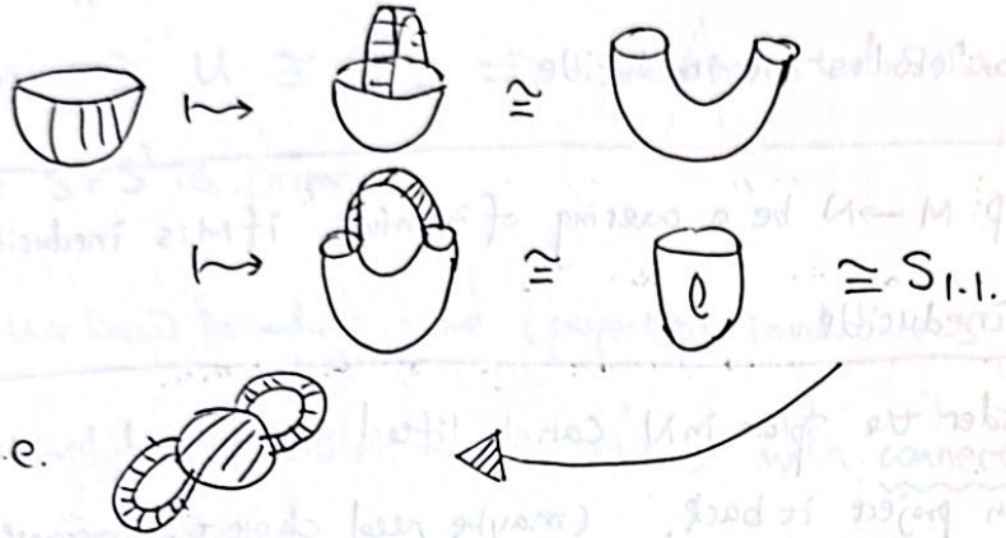
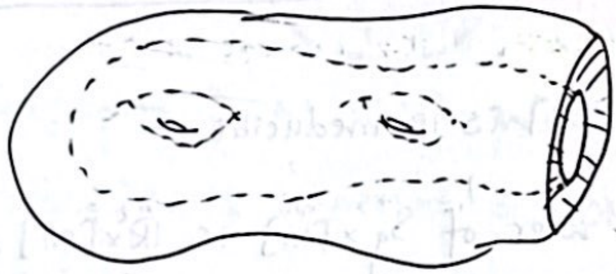
and the genus can be calculated use π_1.

#

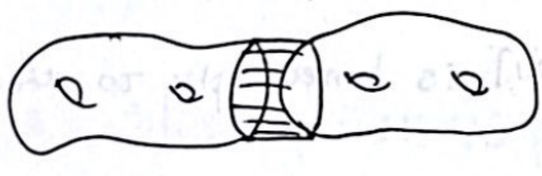
Rmk: ① How to visulize the handle decomposition:



② Another way:



View $S_{g,b}\times[0,1]$ in $\mathbb{R}^3$. 想象从里面"抽出来"或者直接看到两个



把内表面放到外面来.

#

III. Normal surface method & Prime Decomposition Thm

We will mainly use Normal surface theory to give an upper bound for "some special embedded surfaces", hence we can know the end of the prime decomposition.

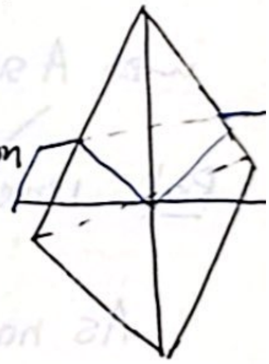
Notation: let M be a compact with (possibly empty) boundary.

- It has a triangulation T.
- T has a certain number t of tetrahedra.

Def: A properly embedded surface $S \subseteq M$ is transverse to T if it is transverse to all its simplexes. In particular, S transverse to T is equivalent to that

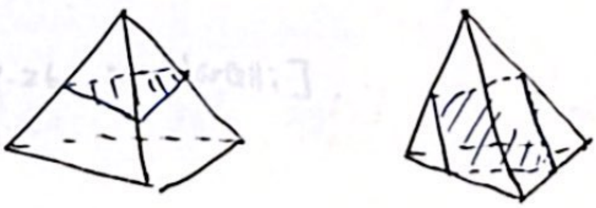
- S does not intersect the vertices of T
- S intersects every edge, face and tetrahedron respectively into a finite number of

points, curves, surfaces

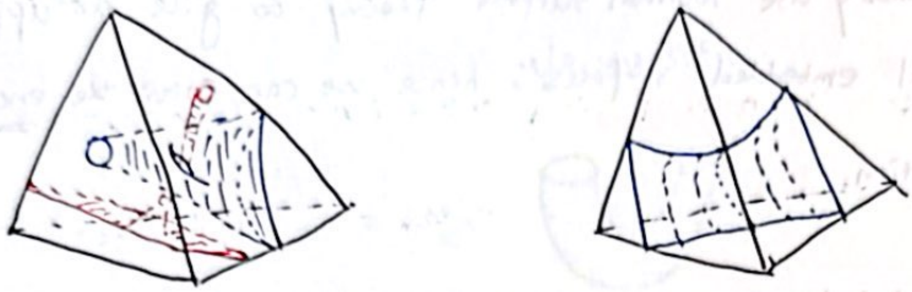


FACT: Every properly embedded surface $S \subseteq M$ can be perturbed to transverse to T.

Def: A normal surface is a properly embedded surface S transverse to T, that intersects every tetrahedron into triangles or squares. i.e.



non-Example (of normal surface)



Example: • For $\forall$ vertex v of T lying in M° we may take a small sphere centered at v that intersects every incident tetrahedron in a small triangle

$\Rightarrow$ we get a normal sphere.

• If $\forall v \in \partial M$. then similarly we get a normal disk.

$\rightsquigarrow$ A surface of this type is called vertex-linking.

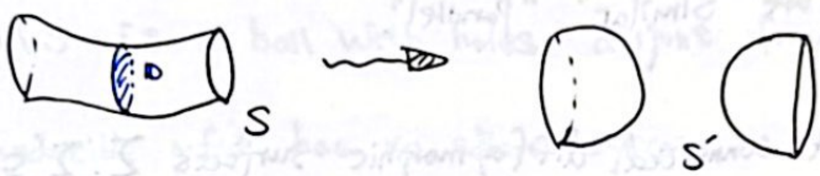
Rmk: vertex-linking sphere are not very interesting since they bound balls

As normal surface is standard (only triangle & square locally) we want to treat each surface into normal after "surgeries".
(Something like elementary transformation is linear algebra)

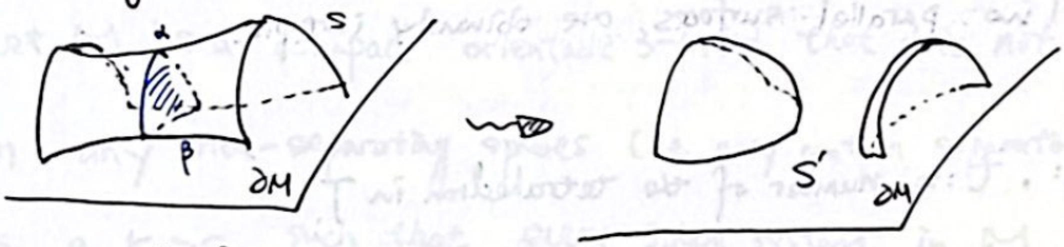
Def: let $S \subseteq M$ be a properly embedded, possibly disconnected, compact surface. An elementary transformation on S is one of the following:

- (E1) the removal of a connected component of S contained in some ball. (it will bound a handlebody. not much interesting)

- (E2) let $D \subseteq M^\circ$ be a disk with $\partial D = D \cap S$. We surger a surface S along a disk D by ① remove an anular tubular nbhd of ∂D in S ② add two parallel copies of D into new S'



- (E3) let $D \subseteq M$ be a disk with $\partial D = D \cap (S \overset{\cup}{\boxminus} \partial M) = \partial \cup \beta$.
Where ∂ is an arc in S and β is an arc in $\beta \partial M$. we do the surgery along D. as



$\alpha \subseteq S$. $\beta \subseteq \partial M$

Prop: Every properly embedded surface $S \subseteq M$ becomes normal after finitely isotopies and elementary transformations.

Hint: Induction on the number of intersections of S with the 1-skelton of T.

[Ref Prop 9.2.26 in Martelli].

• Phenomenon: We have already know that a compact orientable M cannot contain two many disjoint non-orientable surfaces.

⚠: But it can contain arbitorly many orientable surfaces

$\rightsquigarrow$ Chose many small disjoint balls.

However, most of them are not so "interesting", because most of the components are "Similar", "Parallel".

Def: Two disjoint connected diffeomorphic surfaces $\Sigma, \Sigma' \subseteq M$ are parallel if they cobound a region diffeomorphic to $\Sigma \times [0,1]$, with $\Sigma = \Sigma \times 0$ & $\Sigma' \cong \Sigma \times 1$.

Rmk: Two parallel surfaces are obviously isotopic.

Notation: • $t :=$ number of the tetrahedron in T.

• $b_2 := \dim H_2(M, \partial M; \mathbb{Z}_2)$

LEMMA: Let S be an orientable normal surface. If S has more than $10t + b_2$ components, then $\exists$ two components Σ, Σ' of S are parallel, and cobound a $\Sigma \times [0,1]$ which is disjoint from the other components.

Pf: totally combinatorical. see [Mart. Lem 9.2.27]

What is important is the following topological corollary.

Def: • A ball with holes in a 3-mfld obtained by removing some $k \geq 0$ disjoint small open balls from $\mathbb{B}^3$.

• A sphere system for a 3-mfld M is a surface $S \subseteq M^\circ$, consisting of disjoint spheres, s.t. NO components of $M \setminus S$ is a ball with holes disjoint from ∂M.

Rmk: • Irreducible mfld has no sphere system.

• It is hard to construct a "large" sphere system. this can be felt from the defn. hence we can guess there must be an upper bound for the number of spheres in sphere system.

Prop: Let M be a compact orientable 3-mfld that does not contain any non-separating spheres (i.e. only contain separating spheres). There is a $k > 0$ such that every sphere system in M contains less than k spheres.

IDEA: step 1: check that elementary transformation doesn't change sphere system to non-sphere system.

step 2: Assume the sphere system is normal. (WLOG)

step 3: Chose $k \geq 10t + b_2 + 1$. then by lemma above, $\exists S_1, S_2$ cobound $S^2 \times [0,1]$. hence bounds a ball with one hole. ##

Now we can start the proof of Prime decomposition.

Thm (Kneser '29, Milnor '62)

Every compact oriented 3-mfld M (with possibly empty boundary) decomposes into prime manifolds:

$$M = M_1 \# \cdots \# M_k.$$

Rmk: This list of prime factors is unique up to permutations and adding/removing copies of S^3.

Proof: PART I: EXISTENSE of PRIME DECOMP.

- If M contains a non-separting sphere S

$\Rightarrow$ consider the transverse 1-point closed curve δ, and from $N(S) \cup N(\delta)$ we have $M = M' \# (S^2 \times S^1)$ (recall $N(S) \cup N(\delta)$ has sphere boundary, and after capping off, it is $S^2 \times S^1$). $\Rightarrow$ recall $S^2 \times S^1$ is prime

Since $H_1(M) = H_1(M') \oplus \mathbb{Z}$ $\Rightarrow$ up to factoring finitely many copies of $S^2 \times S^1$ we may suppose that every sphere in M is separting.

- Now assume Each sphere in M is separting.

If M is prime than we are done. If not, we decompose $M = M_1 \# M_2$. We keep decomposing each factor until all factors are prime : This process must end, because a decomposition $M = M_1 \# \cdots \# M_k$ give rise to a system of $k-1$ spheres!

✩✩!!

But we already know that $k-1$ can not big from prop. above.

$\# \#$

PART II : UNIQUENESS of PRIME DECOMP.

DEF: We say that a set $S \subseteq M$ of disjoint spheres is a reducing set of spheres, if $\exists$ prime decomposition $M = M_1 \# \cdots \# M_k \#_h (S^2 \times S^1)$ such that $M \setminus S$ consists of precisely one M_i with some holes for $1 \le i \le k$ and some balls with holes disjoint from ∂M.

Example: We may construct a reducing set of spheres of a prime decomp. by chosing the spheres of the decom and one non-separting sphere inside each $S^2 \times S^1$ (this will firstly get Ball with 1 hole then 2 holes).

FACT: • Given Prime decomp $M_1 \# \cdots \# M_k \#_h (S^1 \times S^2)$ and reducing sphere system as above. then add to S any sphere Σ disjoint from S. then $S \sqcup \Sigma$ is still a reducing sphere system for

• We can do surgery for transversely intereseting reducing sphere system. S. S' (for maybe different prime decomp.) such that $S' \mapsto S''$ is reducing sphere system for same prime decom. but $S \cap S'' = \phi$. ([Martelli; Thm 9.2.29])

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Hence, if we have two prime decomp.

$$M = M_1 \# \cdots \# M_k \#_h (S^1\times S^2) = M_1' \# \cdots \# M_{k'}' \#_{h'} (S^1\times S^2)$$

Then for reducing sphere system S, S', from FACT II, WLOG $S \cap S' = \phi$. From FACT I $\Rightarrow$ $S \sqcup S'$ is the reducing sphere system for both prime-decomposition!!

IDEA: 类似于寻找公共加细!! 用 reducing sphere system 来切.

$$\Longrightarrow M|S\cup S' = \begin{cases} \text{Each one } M_i \text{ with holes } + \text{ several balls with holes} \\ \qquad \updownarrow \text{1-1} \\ \text{Each one } M_i' \text{ with holes } + \text{ several balls with holes} \end{cases}$$

$\Rightarrow$ M_i and M_j' are pairwise homeomorphic. $\Rightarrow k = k'$

$\Rightarrow$ $M = N \#_h (S^1\times S^2) = N \#_{h'} (S^1\times S^2) \Rightarrow H_1(M) \cong H_1(N) \oplus \mathbb{Z}^h \cong H_1(N)\oplus \mathbb{Z}^{h'}$

$\Rightarrow h = h'$ $\Rightarrow$ prime decomp is unique. $\square$

Rmk: By cutting along spheres, we have attained a lot of interesting results, so we may turn to do similarly as spheres $\leadsto$ cut along discs.

IV. Essential Disks

Def: Let M be a compact 3-mfld with (possibly empty) boundary.

- A properly embedded surface $S \subseteq M$ is ∂-parallel, if it is obtained by slightly pushing inside M the interior of $S' \subseteq \partial M$ (i.e. $\exists S' \subseteq \partial M$. $\partial S = \partial S'$. and $S \cup S'$ bounds $(S\times I)/\partial S$)
- A properly embedded sphere $S \subseteq M$ is essential, if it does not bound a ball.

• A properly embedded disc $D \subseteq M$ is essential, if it is not ∂-parall.

Def: M is called
- irreducible. if it does not contain essential spheres.
- ∂-irreducible. if it does not contain essential discs.

Prop: Let M be a cmpt. orientable. irreducible 3-mfld with boundary. Let $D \subseteq M$ be an essential disc, $\Sigma \subseteq \partial M$ be the boundary component containing ∂D. Then
- The curve ∂D is non-trivial in Σ.
- If Σ is a torus, then M is a solid torus.

Proof: (1) If ∂D is trivial in Σ. then it bounds a disc in Σ. i.e. D'. then $D \cup D'$ is a sphere in M. Since M is irreducible $\Rightarrow D \cup D'$ bounds a ball $\Rightarrow D$ is ∂-parallel. which is not essential !

(2) Since ∂D is non-trivial. and Σ is a torus $\Rightarrow \Sigma$ cut along ∂D will get a Annuls.

We surgery along D. i.e. remove D and attach two "D". hence we have Σ' is a sphere $\Rightarrow$ bounds a ball. then we know that M is a ball with one handle $\Rightarrow M$ is a solid torus.

$\square$

Rmk: The opposite operation of cutting a mfld along a properly embedded disc is a 1-handle addition.

Hence we can furthermore cut along essential disc and have a more accurate decomp.

「cut along ∂-parallel disc only get ball」

Thm: Every compt oriented irreducible 3-mfld M is obtained by adding 1-handles to a finite list

$$M_1, \cdots, M_k$$

of connected irreducible, ∂-irreducible 3-mflds. The list is unique up to permutations and adding/removing balls.

Example: For ~~unk~~ $K \neq$ unknot. $S^3 \setminus N(K)^\circ$ is irreducible & ∂-irreducible.

- it is irreducible since $S^3 \setminus N(K)^\circ \subseteq S^3$. any sphere bounds two ball in S^3. there must $\exists$ 1 of the two balls $\cap N(K) = \phi$.
- it is ∂-irreducible. this is because $S^3 \setminus N(K)^\circ$ is tori-boundary. but it is not solid torus. so it does not contain essential disk.

Proof: Consider "Disc system" and it cannot have too much discs.

Ref [Martelli Thm 9.2.33]. ##

「Digression:

Prop: Every cmpt, irreducible, orientable 3-mfld that contains $\mathbb{RP}^2$ is diffeomorphic to $\mathbb{RP}^3$ 」

Pf: Since $\mathbb{R}P^3$ is orientable $\Rightarrow$ the tubular nbhd of $\mathbb{R}P^2$ of M is $\mathbb{R}P^2 \tilde{\times} I$

has sphere boundary. it bounds a ball. but $\mathbb{R}P^2 \tilde{\times} I$ has $\mathbb{Z}/2$ as fundamental grp $\Rightarrow$ it is not a ball

$$\Rightarrow M = (\mathbb{R}P^2 \tilde{\times} I) \cup_{S^2} B^3 \cong \mathbb{R}P^3 =$$

□

<u>Exercise</u> (2024 Alibaba)

M is an orientable irreducible 3-mfld contains $\mathbb{R}P^2 = N_1$. then M contains N_g ($\#_g \mathbb{R}P^2$) iff g is odd.

Pf: $N_g = \mathbb{R}P^2 \# (g-1) T^2$

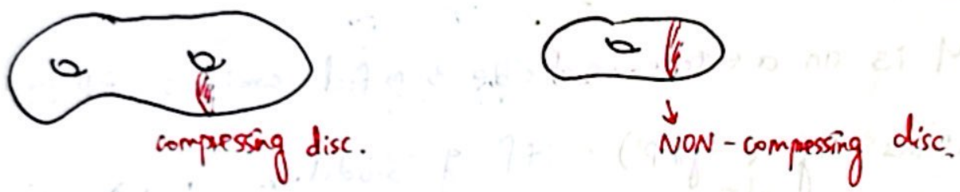
Easy to see $M \cong \mathbb{R}P^3$.

§ 9.3 Incompressible Surfaces

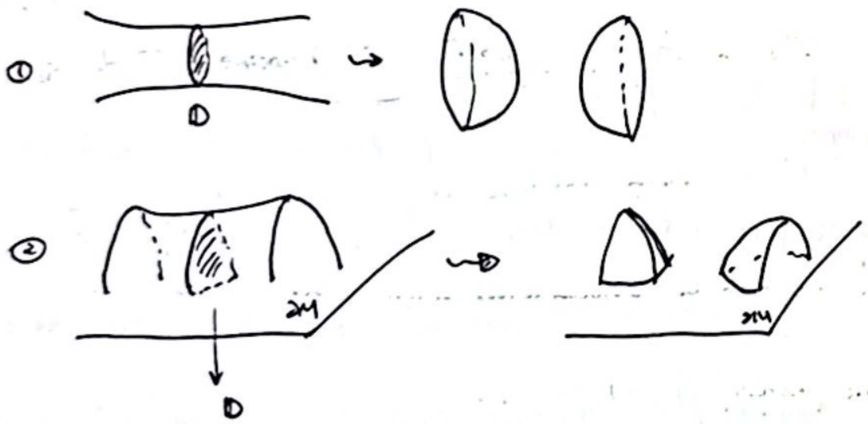
Def: Let M be a cmpt orientable 3-mfld with (possibly empty) boundary. Throughout this section

Let $S \subseteq M$ be a properly embedded orientable surface.

- A compressing disk for S is a disc $D \subseteq M$ with $\partial D = D \cap S$ such that ∂D does NOT bound a disc in S.

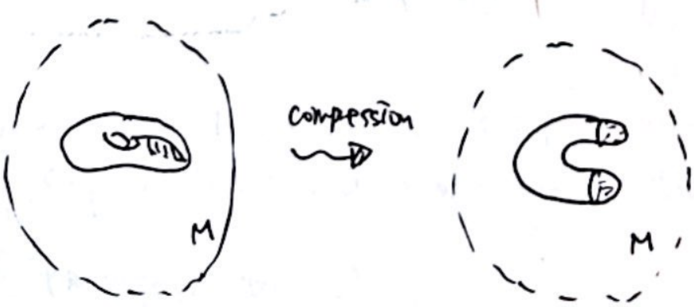


- A compression is a surgery along D. as



A compression makes S to S'.

Prop: After compression, S' may have two components /or 1 component S_i'. and $\chi(S_i') > \chi(S)$ for each component.



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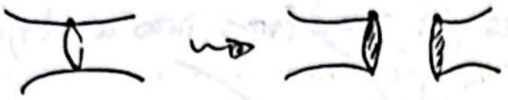
Proof: Note that $\chi(S') = \chi(S) + 2$



$V' = V+1$
$E' = E+1$
$F' = F+2$

- If $\pi_0(S') = 1$, then we finished.

- If $\pi_0(S') = 2$. and $S' = S_1' \sqcup S_2'$ $\Rightarrow$ $\chi(S_1') + \chi(S_2') = \chi(S) + 2$.

Claim: No S_i' is a sphere. if S_i' is a sphere. then S is

$\Rightarrow$ ∂D bounds a disc on S $\Rightarrow$ D is not a compressing disc.

$\Rightarrow \chi(S_i') \leq 1$ $\Rightarrow$ we finished.

##

Rmk: The prop above tells us that compression will increase χ of surface

Def: A properly embedded cco surface $S \subseteq M$ with $\chi(S) \leq 0$ is called compressible, if it has a compressing disc (hence can do compression).
And it is called incompressible. if it has no compressing disc.

Example: $M = T^2 \times [0,1]$. $S = S^1 \times [0,1]$ the annulus.

is an incompressible surface.
(가ㅋ compressing Disc)

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Prop: Let $S \subseteq M$ be any properly embedded orientable surface. After compressing it a finite number of times it transforms into a disjoint union of spheres, discs and incompressible surfaces.

Proof: Finiteness from $\chi(S') > \chi(S)$ estimation. ##

Rmk: Essential disc vs compressing disc.

Essential disc: For 3-mfld M. not ∂-parallel.

Compressing disc: For surface $S \subseteq M$. ∂D not bounds disc.

$\downarrow$ $\qquad$ $\downarrow$

∂-irreducible vs incompressible surface

∂-irreducible: For 3-mfld M NOT Has essential disc

Incompressible surface: For surface $S \subseteq M$ NOT has compressing disc.

常用

Then consider the boundary of an ∂-irreducible mfld M. then $\forall$ component S of ∂M is incompressible.

不∃本质圆盘 $\Longleftrightarrow$ 不∃压缩圆盘

Prop: Let $S \subseteq M$ be an orientable, connected, properly embedded surface with $\chi(S) \leq 0$. If $i_*: \pi_1(S) \to \pi_1(M)$ induced by inclusion is injective $\Rightarrow$ S is incompressible.

Proof: This is Easy. If S is compressible $\Rightarrow$ $\exists$ compressing disc $D \subseteq M$. s.t. $\partial D = D \cap S$. and ∂D does NOT bound disc on S

$\Rightarrow$ $[\partial D]$ is non trivial in S. but trivial in M !! $\square$

$\boxed{\text{FACT: Actually: } S \text{ is incompressible} \Leftrightarrow i_*: \pi_1(S) \to \pi_1(M) \text{ embedding}}$

$\Leftarrow$ nontrivial !! (proof later)

Prop: If $S \subseteq M$ is incompressible, then every component of ∂S is non-trivial in ∂M.

Proof: If $\gamma \subseteq \partial S$ is trivial in ∂M. then it bounds a disc. D. pushing it inside by collar thm. we get a compressing disc for S. □

I. Tori in 3-mfld

> **Prop**: Let $T \subseteq M$ be a torus in an irreducible 3-mfld. one of the following holds:
>
> (1) T is incompressible.
>
> (2) T bounds a solid torus.
>
> (3) T is contained in a ball.

Proof: If T is compressible $\Rightarrow$ $\exists$ a compressing Disc. D.

compression along D $\rightsquigarrow$ T' is a sphere. in M. hence bounds a ball. B

Either

- T disjoint with B ↘ 内部补实球 →

or

- T contained in B → 外部补实球 → (3).

□

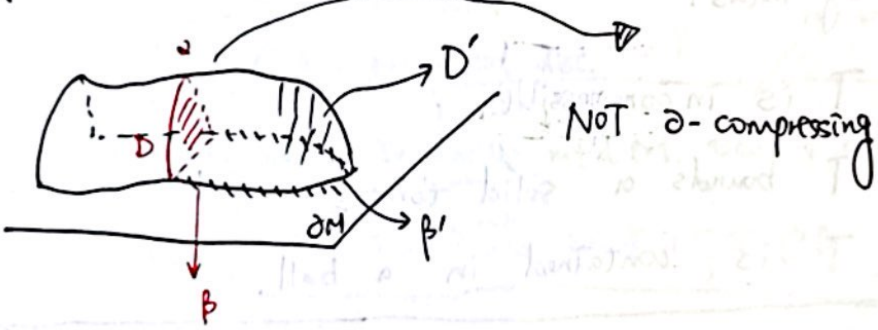
Corollary: The torus T in S^3 must bound a handlebody !!

Proof: – $\pi_1(S^3)=0 \Rightarrow T$ cannot be incompressible.

– If T is contained in a ball of S^3, then it bounds a solid torus on the other side. $\square$

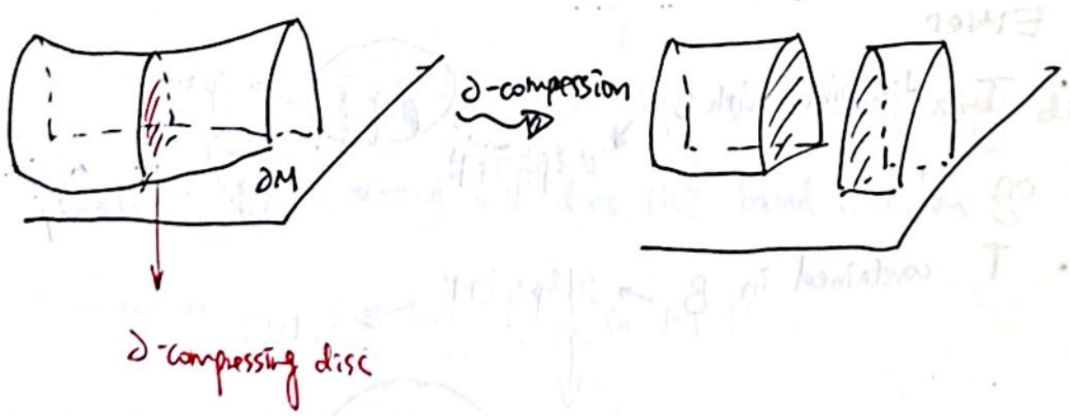
Def: $S \subseteq M$ is a properly embedded orientable surface.

- A ∂-compressing disc for S is a disc D with $\partial D = \alpha \cup \beta$ where $\alpha \subseteq S$. $\beta \in \partial M$. s.t. 不 $\exists$ $\beta' \in \partial M$. and disc $D' \subseteq M$. s.t. $\partial D' = \alpha \cup \beta'$.



i.e.

- A ∂-compression is similar as compression

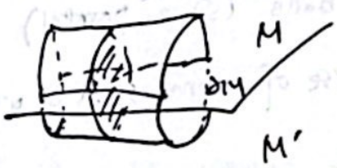


Rmk: 要求 ∂-compression disc 只是希望沿着 disc 作 ∂-compression 后, 不会得到平凡的如 disc

Def: $S \subseteq M$ properly embedded with $\chi(S) \le 0$. is called ∂-compressible if $\exists$ ∂-compressing disc. If NOT exists, then called ∂-incompressible.

Prop: $S \xrightarrow{\partial\text{-compression}} S'$. then $\chi(S') > \chi(S)$. / $\chi(S_i') > \chi(S)$.

Proof: Double M along ∂M. then from $\chi(\tilde{S}') = \chi(\tilde{S}) + 2$ for compression version $\Rightarrow$ $\chi(S') = \chi(S) + 1$. then similarly.



□

Corollary: After finite-time ∂-compression. $S \rightsquigarrow$ disjoint union of spheres, discs and ∂-incompressible surfaces.

II. Annuli in 3-mfld.

Prop: Let $A \subseteq M$ be a propely embedded annulus in irr. and ∂-irr M, one of the following holds:

(1) A is incompressible and ∂-incompressible

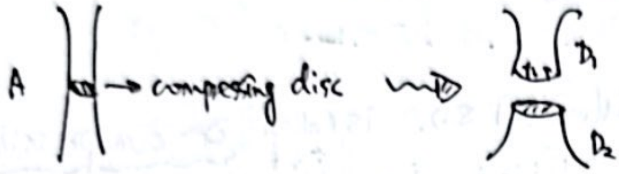
(2) A bounds a tube.

(3) A is parallel to an annulus in ∂M.

(4) A is contained in a ball B intersecting ∂M in a disc.

Proof: We assume (1) does NOT hold

Case I: If A is compressible along a disc $\mathbb{D}$. then after compressing

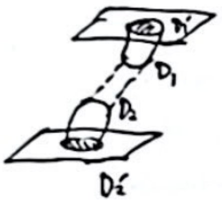


transforms into two discs. D_1. D_2.

Note that M is ∂-irr. and ∂D_1. $\partial D_2 \subseteq \partial M$. (since $\partial D_1 \sqcup \partial D_2 = \partial A \subseteq \partial M$).

then we know that M does not contain essential disc. i.e. all disc are ∂-parallel. $\Rightarrow \exists\ D_1'.\ D_2' \subseteq \partial M$. parallel to D_1. D_2 resp.

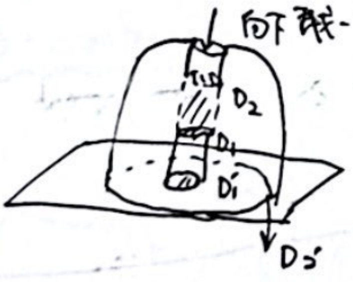
- If $D_1' \cap D_2' = \phi$, then consider $S_1 = D_1 \cup D_1'$. $S_2 = D_2 \cup D_2'$ are sphere bound balls (since parallel)



note that reverse of compression is attach 1-handle

$\Rightarrow$ we get a tube. with bounded by A. $\Rightarrow$ (2)

- If $D_1' \cap D_2' \neq \phi$ $\Rightarrow D_1' \subseteq D_2'$ wloG.



then Annulus is contained in the ball bounded by D_2' and D_2. $\rightarrow B$

and $B \cap \partial M = D_2$ is a disc. $\Rightarrow$ (4)

Case II: If A ∂-compressible along a disc D.

then

$\Rightarrow$ cuting along $\rightsquigarrow$ get a disc bound a ball since irred. $\Rightarrow$ parallel to boundary $\Rightarrow$ (3)

$\square$

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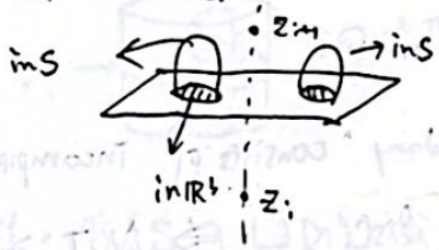
Rmk: 简记. 我们希望记住. 对于 3-mfld 中的正则嵌入 torus & Annulus. 我们有十分标准的刻画!! (完全分类).

Now. Let's see the surfaces in special 3-mflds.

Prop: There are no incompressible surfaces in $\mathbb{R}^3$.

Proof: If S is an incompressible surface in $\mathbb{R}^3$. rotate properly such that height function is Morse. and critical points $z_1, \dots, z_k$ are different. $z_1 < \dots < z_k$.



Each plane cut S into several discs. trivially it bounds disc in $\mathbb{R}^3$. since incompressible $\Rightarrow$ this discs bound disc in S. $\Rightarrow$ after compressing. we will get several disjoint spheres

$\rightsquigarrow$ S is attach 1-handle to $\underset{\text{several.}}{\sqcup} S^2$. not attah on the same one. i.e. 任意两个的连通和 $\Rightarrow$ $S \cong S^2$ or $\underset{\text{若干}}{\sqcup} S^2 \Rightarrow \chi(S) > 0$. contradicting to S incompressible. $\square$

Corollary: There are no incompressible surface in S^3.

Corollary: There are no incompressible surface in $\mathbb{B}^3$.

All can also easily seen from π_1 trivial

Corollary: • Every torus in S^3 bounds a solid torus ;

• Every properly embedded Annulus in B^3 bounds a tube .

Sketch: Easily from the classification thm of tours & Annulus

Prop (For Handlebodies)

Let H_g be the genus g handlebody. then

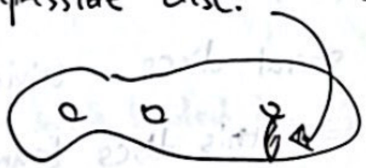
(1) H_g is irred. but NOT ∂- irred.

(2) H_g contains NO (incompressible + ∂- incompressible) surfaces.

Proof: (1) H_g is irred. from $\pi_2 = 0$.

Recall ∂- irred. $\Leftrightarrow$ boundary consists of incompressible surfaces.

but $\partial H_g = S_g$ has compressible disc. $\Rightarrow$ NOT ∂- irred.



(2) Suppose $S \subseteq H_g$ is incompressible and ∂-incompressible.

Consider $D_1, \cdots, D_g$ as (Take $g=2$ for example)

Cut along D_i $(1 \leq i \leq g)$ $\rightsquigarrow$ get a ball.

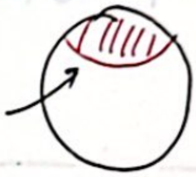
Suppose $S \pitchfork \bigsqcup_{i=1}^{g} D_i$.

• If $S \cap \bigsqcup_{i=1}^{g} D_i = \phi$ $\Rightarrow$ S is incompressible and ∂-incompressi in the ball. but B^3 contains NO incompressible surface $\Rightarrow$ 矛盾!

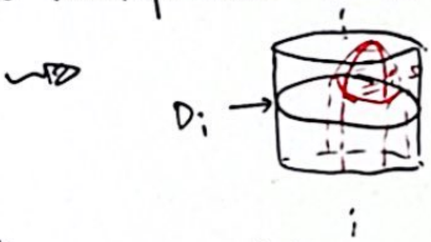
• If $S \cap \bigsqcup_{i=1}^{g} D_i \neq \emptyset$, then at each D_i looks like.

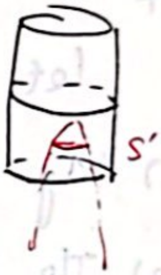
$\to S \cap \bigsqcup D_i$: several arcs / several discs.

D_i

•• For inner-most arcs, it bounds disk with  . since

S is ∂-incompressible $\Rightarrow$ it bounds another disc.



S可沿disc向下 isotopy $\rightsquigarrow$ 

从而 经过 isotopy, $\#(S \cap \bigsqcup D_i)$ 减少.

•• For inner-most discs, similarly from incompressible. 

向下推 isotopy

$\rightsquigarrow \#(S \cap \bigsqcup D_i) \downarrow$

Finally after finite remove. we get $S' \cap \bigsqcup_{i=1}^{g} D_i = \emptyset$. and S' is isotopy with S. hence also incompressible & ∂-incompressible. then from the first case $\Rightarrow$ 矛盾!

$\square$

<u>Rmk</u>: especially, 柄体中不可压缩曲面一定带边!!

Prop: The incompressible surface in solid torus H_1 can only be ∂-parallel Annulus.

Proof: [Ref. Martelli 9.3.16].

Rmk: H_1 中不可压缩曲面简单是因为其基本群小. 但对 H_g. $(g\geq 2)$. 基本群丢大丢可以包含任意秩的 free grp. $\Rightarrow$ 其中不可压曲面可以很多杂!!!

Prop: (For line bundles)

Let $M_g = S_g \times [-1,1]$, then

(1) M_g is irred. and ∂-irred;

(2) The incompressible and ∂-incompressible surfaces in M_g are

- the horizontal surface $S_g \times \{0\}$
- A vertical annulus $r \times [-1,1]$. for each non-trivial simple closed curve $r \subseteq S_g$.

Proof: (1) $\pi_2(M) = 0 \Rightarrow M$ is irred.

$\pi_1(S_g \times \{\pm 1\}) \hookrightarrow \pi_1(S_g \times [-1,1])$ is embedding $\Rightarrow \partial(S_g \times [-1,1])$ all incompressible $\Rightarrow S_g \times [-1,1]$ is ∂-irred.

(2) see [Martelli 9.3.18].

##.

Notation Summary:

$\to \pi_2$

① irreducible. / essential sphere.

∂ boundary π_1 检验. $\Rightarrow$ boundary incompressible.

② ∂-irreducible / essential disc. / ∂-parallel.

③ compressible disc. / compression / incompressible surface. $\to \pi_1$

④ ∂-compressible disc / ∂-compression / ∂-incompressible surface

§ 9.4 Haken Manifolds

Before we start. Let's summarize the def appears in previous sections:

- $\boxed{\text{Essential sphere}}$: that does not bound ball
- $\boxed{\text{Essential disc}}$: that does not ∂-parallel.

- $\boxed{\text{irreducible}}$: 3-mfd NOT has essential sphere. ($\Leftrightarrow \pi_2 = 0$)
- $\boxed{\partial\text{-irreducible}}$: 3-mfd NOT has essential disc. ($\Leftrightarrow$ boundary all incomp.)
- $\boxed{\text{incompressible surface}}$: $S \subseteq M$. NOT has compressing disc — π_1 embedding
  $\chi(S) \leq 0$ (i.e. 不能作非平凡 compression 操作) → 切出 sphere
- $\boxed{\partial\text{-incompressible surface}}$: $S \subseteq M$. NOT has ∂-compressing disc
  (i.e. 不能作非平凡 ∂-compression 操作). → 切出 disc.

The defns above are all important. so we collect them and get:

I. Haken mfds & their embedded surfaces

Def: A Haken mfd is a CCO 3mfd with (possibly empty) boundary, which is

- irreducible,
- ∂-irreducible,
- contains an (Incompressible and ∂-incompressible) surface.

只含一个即可

同时满足!!!

Rmk: The reader should not be frightened by the abundance of adj. this defn is really clever because it summarizes various reasonable hypothesis in a unique word.

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Rmk: Some quick criterion review

- irreducible : $\pi_2 = 0$
- ∂-irreducible : all boundary surfaces are π_1 embedding.
- incompressible : π_1 embedding (Prove latter).
- If $\partial M = \phi$, then no need to consider ∂-irrd / ∂-incompe conditions!

Examples & Non-Examples

1) $\mathbb{R}^3$, S^3, $\mathbb{B}^3$ are NOT Haken mfids.

(recall in last section, all does NOT contain incompressible surface).

2) Handlebody are NOT Haken mfids

- Not ∂-irred : $\exists$ essential disk 

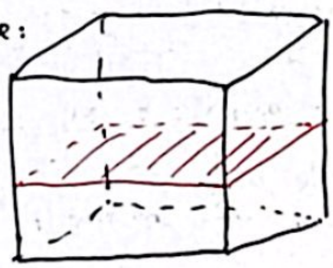
or not π_1-embedding ($\pi_1(S_g)$ has relations).

- Not have (incompres & ∂-incomp) surface.

(last time proved).

3) $T^3 = S^1 \times S^1 \times S^1$ IS Haken mfid.

- irred : $\pi_2 = 0$ / covering $\mathbb{R}^3$ is irred.
- incompressible surface :



standard $T^2 \hookrightarrow T^3$

(π_1- embedding)

- $\partial T^3 = \phi$. no need to consider ∂-irred / ∂-incompressible.

4) line bundle : $S_g \times [-1,1]$ Is Haken mfid.

(last time proved). (incomp and ∂-incompe) $\begin{cases} \text{horizontal} & S_g \times \{0\} \\ \text{vertical} & r \times [-1,1] \end{cases}$

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<u>Rmk</u>: Actually, Haken mflds are a lot. we will see latter.

Now we will study Haken mflds through their embedded surfaces.

Prop: Every boundary component X of a Haken mfld M has $\chi(X) \le 0$ and is incompressible.

<u>Proof</u>: Well-known that Haken mfld is ∂-irred $\Rightarrow$ boundary consists of spheres or incompressible surfaces. But if $\exists\, X = S^2$, then from M is irreducible $\Rightarrow$ M is a ball B^3. but B^3 is not Haken. contradiction $\square$

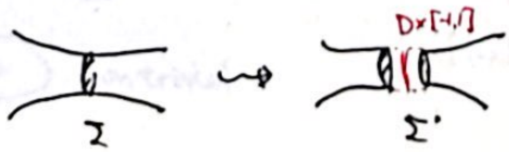
We now prove that there are plenty of Haken mflds, we start with a general proposition: (找 Haken 流形 困难从 irre/及 ∂-irred/ 出发. 找 incomp. 曲面, 下面命题给出了个构造不可压缩曲面的方案).

Prop: Let M be a cco, irreducible, ∂-irred, 3-mfld, with (possibly empty) boundary. <u>Then</u>: Every non-trivial homology class $\alpha \in H_2(M, \partial M; \mathbb{Z})$ is represented by a disjoint union of (incomp & ∂-incomp) oriented surfaces.

<u>Proof</u>: • Recall: $\forall\, \alpha$ can be represented by a properly embedded oriented surface S (Hint: $H_2(M, \partial M) \cong H^1(M) \cong [M, S^1]$);

• Recall: after finite time of compression and ∂-compression, we will transform S into S' which is <u>a disjoint union of discs, spheres and (incomp & ∂-incomp) surfaces</u>.

•• Easy to see compression & ∂-compression do NOT alter the homology class:



$\Sigma - \Sigma' = \partial(D \times [-1,1])$

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$\Rightarrow [S] = [S']$

.. since M is $\begin{cases} \text{irred} \Rightarrow \text{sphere bounds ball} \Rightarrow \text{homological trivial} \\ \partial\text{-irred} \Rightarrow \text{discs all } \partial\text{-parallel} \Rightarrow \text{homological trivial.} \end{cases}$

(相对也行，因此消失).

$\Rightarrow$ remove discs & spheres in $S' \rightsquigarrow S''$. which is a disjoint union of (incomp & ∂-incomp) surfaces. and

$\alpha = [S] = [S'] = [S'']$. □

$$\boxed{\text{corollary: let } M \text{ be a cco, irred, } \partial\text{-irred 3-mfld. if } H_2(M, \partial M) \neq 0, \text{ then } M \text{ is Haken.}}$$

corollary: If $M = \mathbb{H}^3/\Gamma$ is a closed orientable hyperbolic mfld. with $\dim_{\mathbb{Z}} \Gamma^{ab} \neq 0$, then M is Haken.

Proof: M hyperbolic $\Rightarrow$ irred. $\dim_{\mathbb{Z}} \Gamma^{ab} \neq 0 \Rightarrow H_1(M)$ free part $\neq 0$

$\Rightarrow H_2(M) \neq 0 \Rightarrow M$ is Haken. □

$$\boxed{\text{corollary: let } M \text{ be a cco, irred, } \partial\text{-irred 3-mfld. if } \partial M \neq \emptyset \text{ \& } M \neq B^3, \text{ then } M \text{ is Haken.}}$$

Proof: $M \neq B^3 \Rightarrow \partial M$ cannot contain sphere hence $H^1(\partial M)$ has positive rank $\Rightarrow H_2(M, \partial M) \cong \cancel{H^1(\partial M)}\, H^1(M)$, then recall $b_1(M) \geq \frac{1}{2} b_1(\partial M) > 0$

$\Rightarrow H_2(M, \partial M) \neq 0 \Rightarrow M$ is Haken. □

Rmk: We recall that every cmpt orientable 3-mfld decomposes along essential sphere/discs into irreducible and ∂-irreducible pieces. Then if one such piece has non-empty boundary then either it is a ball, or it is a Haken mfld.

As we have shown that Haken mflds are a lot, we can now study the surfaces in them. Firstly, we want to study their (incompressible & ∂-incomp) surface, However, through all inner such surfaces, $\exists$ a more special & interesting one — "spanning" surface that touches all the boundary components:

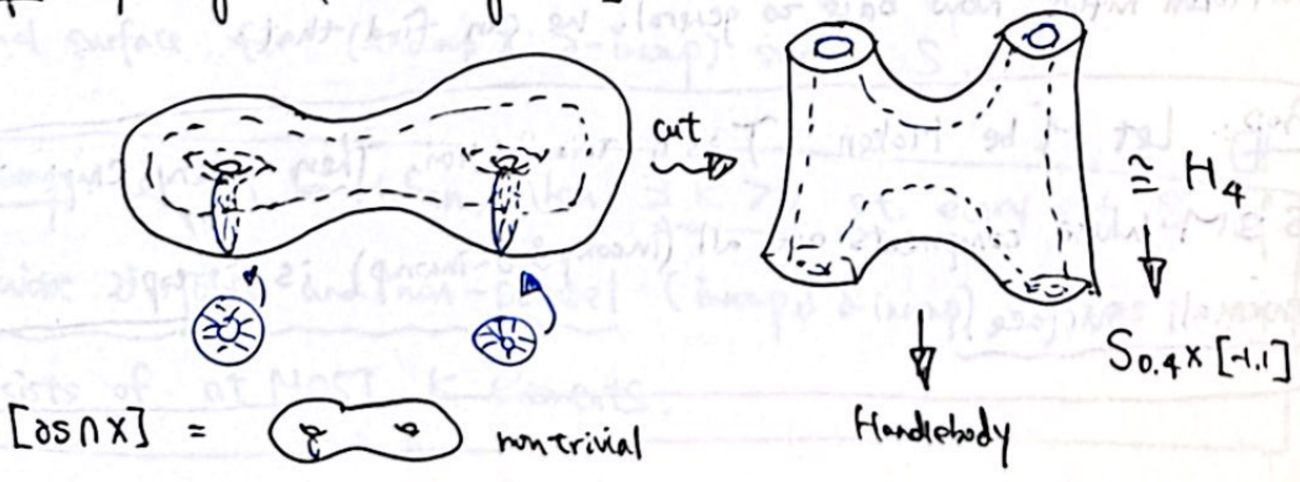
→ spanning surface 存在性.近似对同调类中元素提高要求.

LEMMA: Every Haken mfld M contains an oriented surface S, whose components are ① (incomp & ∂-incomp), such that for $\forall$ boundary component X of M, ② $[\partial S \cap X] \in H_1(X;\mathbb{Z})$ is non-trivial.

分层结构/统治集团 ニ

Rmk: In the future hierarchy, we will cut M along the spanning surface to get a handlebody!

Example: spanning surface in $S_g \times [-1,1]$



Proof of lemma: We have $\partial M = X_1 \sqcup \cdots \sqcup X_k$, where $\forall i$. $\chi(X_i) \le 0$ and X_i is incompressible (from previous discussion)

$$\Rightarrow H_1(\partial M; \mathbb{Z}) \cong \bigoplus_{i=1}^{k} H_1(X_i; \mathbb{Z}).$$ consider $\partial: H_2(M, \partial M) \to H_1(\partial M)$ with image $L = \mathrm{Im}\,\partial$. we have proved that L is Lagrangian, for $H_1(\partial M)$.

For $\forall \alpha \in L$. we have decomposition $\alpha = \partial_1 + \cdots + \partial_k$. $\partial_i \in H_1(X_i)$.

Claim: $\forall i$. $\exists \alpha$. s.t. $\partial_i \neq 0$. $\left(L|_{H_1(X_i)}\right.$ 为 Lagranigian. 也就是可以$\left.\right)$

If NOT. then $\exists i$. $\forall \alpha$. s.t. $\partial_i = 0 \Rightarrow L$ is ω-orthogonal to $H_1(X_i)$. because $\forall \beta_i \in H_1(X_i)$. $\omega(\alpha, \beta_i) = \omega(\partial_i, \beta_i) = \omega(0, \beta_i) = 0 \Rightarrow L$ is contained in $H_1(X_i)^{\omega}$, the symplectic complement of $H_1(X_i) \Rightarrow L$ is Lagrangian for $H_1(X_i)^{\omega} \Rightarrow \dim L = \frac{1}{2}\dim H_1(X_i)^{\omega} < \frac{1}{2}\dim H_1(\partial M)$. since $\dim H_1(X_i) > 0$ from $\chi(X_i) \le 0$. a contradition!

Hence for such $\alpha \in L = \mathrm{Im}\,\partial$. consider $\alpha = \partial[S]$, then S is desired if we do the same as previous prop. then $[\partial S \cap X_i] = [\alpha \cap X_i] = [\partial_i] \neq 0 \Rightarrow$ as desired. $\square$

Above discussion tells us there are special (incomp & ∂-incomp) surface in Haken mflds. Now back to general. we can find that:

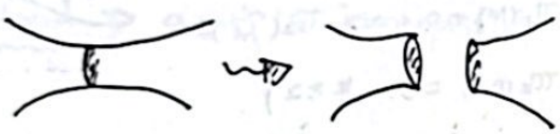
Prop: Let M be Haken. T is a Triangulation, Then Every Cmpt surface $S \subseteq M$ whose components are all (incomp & ∂-incomp) is isotopic to the normal surface.

Rmk: Normal surface 基本上就是某种3-mfd中的标准模型，上述性质实际上在说 不可压缩曲面 同痕一下就变得很标准，可被为手判别与控制.

PROOF: Recall each properly embedded surface can transformed into normal through elementary transformation:

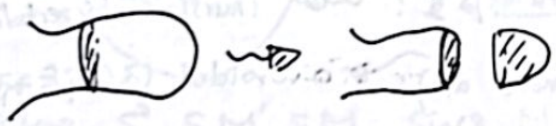
(E1) removal of the components that contained in the ball.

- since B does not contain incomp surface. this case will not happen.

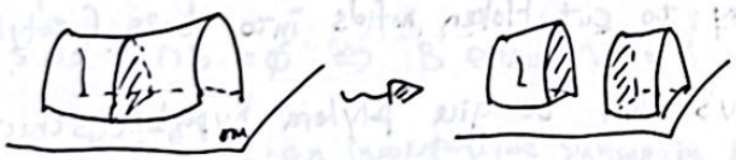
(E2) 

- since S is incompressible $\Rightarrow$ no compressing disc

$\rightsquigarrow$ this transformation only happen as

 i.e. isotopy.

本质: (E2), (E3) 对 incomp & ∂-incomp 曲面 实为同痕.

(E3) 

- since ~~∂~~ S is ∂-incomp $\Rightarrow$ similarly isotopy.

Thus NO (E1). and (E2), (E3) all isotopy $\Rightarrow$ we get a normal surface for (incomp & ∂-incomp) surface S. $\square$

> **Corollary**: Let M be Haken, Then $\exists\, k > 0$ s.t. every set S of pairwise disjoint and non-parallel (incomp & ∂-incomp) surfaces in M consists of at MOST k elements.

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II. Cutting along Surfaces & Hierarchies

注: 本节 we focus on 沿着 (incomp & ∂-incomp) surface 切开 Haken mfld, 并证明, 它们切割有限次后, 我们会获得若干球. (Hierarchy)

Hierarchy 会帮助我们证明许多关于 Haken 流形的基本结果 如:

- Haken 3-mfld is $K(\pi,1)$ space.

(incomp surface existence $\Rightarrow \pi_1(M)$ is infinite $\Rightarrow \widetilde{M}$ (universal cover) non-cmpt. $\Rightarrow H_k(\widetilde{M})=0$.

$k\geq 3$. Since M irredu. $\Rightarrow \pi_2(M)=0 \Rightarrow \pi_2(\widetilde{M})=0 \longleftrightarrow$ By Hurwicz

we have $\pi_i(\widetilde{M})=0 \Rightarrow \pi_k(M)=0$. $k\geq 2$).

- Homotopy equivalent Haken 3-mfld are homeomorphic.

Note: 上述两个结果是 hyperbolic mfld 的基本结果 (Mostow rigidity), 这会启发我们是否 Haken mfld 可以赋予双曲度量? $\leadsto$ Thurston's hyperbolization for Haken mfld: (not all! but almost all) $\to$ atoroidal. (不含本质环面)

step1: Use hierarchy to cut Haken mflds into balls (polyhedra)

step2: use Andreev's thm to give polyhedra hyperbolic structure.

step3: Use skinning lemma to glue them back.

类比 hyperbolization for S_g ($g\geq 2$)

- cut into pair of pants
- hyperbolic str for hexagon $\leadsto$ pair of pants
- glue pair of pants back $\leadsto$ Teichmüller.

简记: 上述给了我们研究并证明 hierarchy 的动机.

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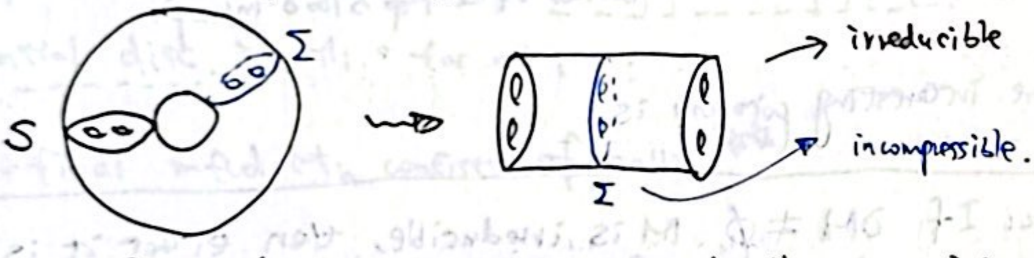
First of all, we prove that cut a 3-mfld along an incompressible surface, some nice properties of the mfld are preserved.

Prop: Let M be cmpt & irreducible 3-mfld. $S \subseteq M$ be either an essential disc or an incompressible surface. Let M' be obtained by cutting M along S. i.e. $M' = M \setminus \mathring{N}(S)$. Then the following holds:

(1) M' is also irreducible;

(2) A closed $\Sigma \subseteq M'$ is incompressible in M' $\Leftrightarrow$ it is so in M.

PROOF: Example: cut $S_g \times S^1$ along $S_g \times \{0\}$



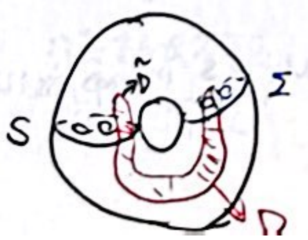
(1) $\forall$ sphere $S^2 \subseteq M' \subseteq M$. since M is irreducible $\Rightarrow$ S^2 bounds a ball B. since $S^2 \cap S = \phi$ $\Rightarrow$ B either $\cap S = \phi$ or contains S. but if the latter, then S is an incompressible surface in ball B. contradiction!

(2) We prove that: $\forall$ closed Σ in M'. Σ has a compressing disc in M' $\Leftrightarrow$ Σ has a compressing disc in M.

($\Rightarrow$) trivial since M' is contained in M.

($\Leftarrow$) Suppose Σ has a compressing disc in M. i.e. D. to find a compressing disc D' in M'. it suffices to surgery D s.t. $D' \cap S = \phi$.

hand disc $\tilde{D}$.



consider the innermost circle of $D \cap S$. since S is ~~not~~ incompressible $\Rightarrow$ we can isotopy s.t. cancel the innermost circle $\Rightarrow \cdots \overset{\text{isotopy}}{\Rightarrow} D' \cap S = \phi$. $\square$

Corollary: If we cut a Haken 3-mfld along a closed incompressible surface, we get a disjoint union of Haken 3-mflds.

Proof: Consider M is Haken. $S \subseteq M$ is a closed incomp. M' is M cut along S. then $M' = M \setminus \mathring{N}(S)$

- M' is irreducible $\Leftarrow$ last prop.
- M' is ∂-irreducible $\Leftarrow$ ($\partial M'$ 比 ∂M 多了一份/两份 S. 由 last prop. S 在 M incomp $\Rightarrow$ S 在 M' 中也 incomp. $\Rightarrow$)
- M' contains (incomp & ∂-incomp) surface) $\Leftarrow$ 即 consider S 即可.

  ↑ closed 保证不用考虑 ∂-incomp.

$\square$.

An more interesting corollary is:

Corollary: If $\partial M \neq \emptyset$, M is irreducible, then either it is a handlebody or it contains a closed incompressible surface. (结构定理)

Rmk: 如果 Haken流形定义中去掉 ∂-irred, 则上述命题告诉我们: 若 3-mfld $\partial M \neq \emptyset$ 且不可约, 则其即为 Haken 流形!!

Proof: After cutting along essential discs, we get $M_1, \dots, M_k$. all irred and ∂-irred.

「recall 切 essential disc 逆操作是 粘1-handle. 」

Case I. If all $\partial M_i = S^2$ $\Rightarrow$ all M_i are ball. then M is handlebody!

Case II. If $\exists$ $\partial M_i = S_g$. $g \geq 1$. then since M_i is ∂-irred $\Rightarrow$ ∂M_i is incompressible in M_i. from previous prop $\Rightarrow$ ∂M_i is incompressible in M. which is desired.

$\square$

Now let's start to talk about hierarchy (分层结构/种姓制度). this is a procedure that use incomp surface to cut Haken mfld into simpler pieces. finally, to balls.

<u>Def</u>: A <u>hierarchy</u> for a Haken 3-mfld M is a sequence of 3-mflds

$$M=M_0 \xrightarrow{S_0} M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} M_n$$

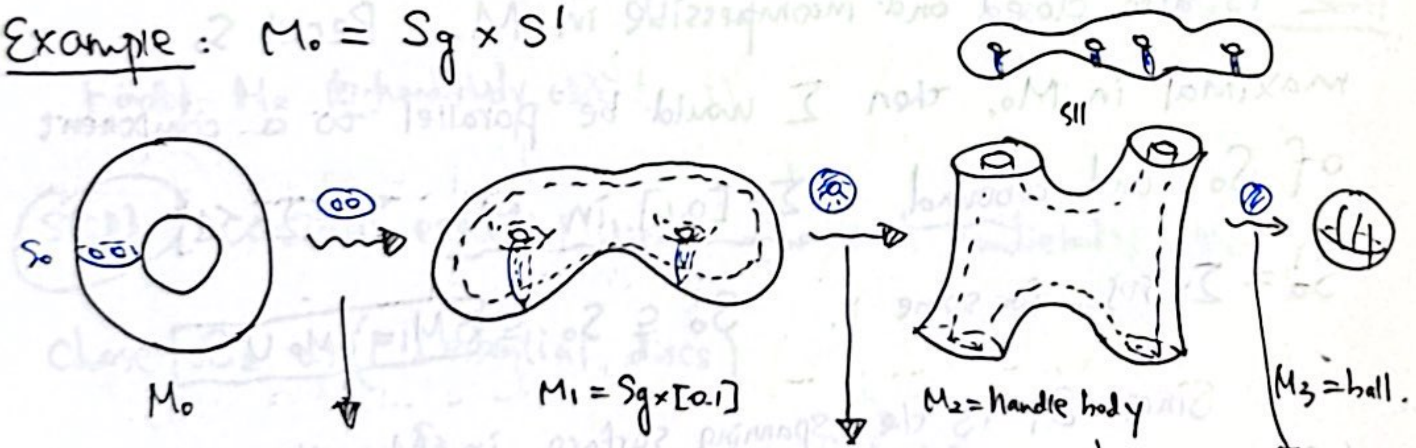
where each M_{i+1} is obtained by cutting M_i along a properly embedded (possibly disconnected) surface $S_i \subseteq M_i$ such that the following holds:

- Every component of S_i is an (incomp & ∂-incomp) surface or an essential disc in M_i. for all i;
- The final mfld M_n consists of balls. (★★)

And the number n is called the <u>height</u> of the hierarchy.

$\boxed{\text{Thm: Every Haken mfld has a hierarchy of height 3.}}$

<u>Example</u>: $M_0 = S_g \times S^1$



沿着极大不可压缩曲面系统

沿着 spanning surface 切开

永远是柄体

沿着本质圆盘.

[step1 去掉了 maximal ⇒ 因为 $\partial \neq \emptyset$, 无 closed incomp surface!]

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Proof: Start with Haken mfld $M = M_0$ 知道能保证有限个的 maximal 一直保证，由 normal surface 理论

(step 1) consider $S_0 \subseteq M_0$, the <u>maximal</u> family of pairwise disjoint and non-parallel closed incompressible surfaces in M. Then <u>cut M_0 along $S_0 \rightsquigarrow M_1$</u>.

沿着不可压缩曲面切开

(step 2) Since every componet M_1^i of M_1 is Haken, then we choose the spanning surface $S_1^i \subseteq M_1^i$ made of (incom & ∂-incomp) surfaces that intersect every boundary component of M_1^i <u>non-trivially</u>. Then <u>cut M_1 along $S_1 = \bigsqcup_i S_1^i \rightsquigarrow M_2$</u>

(结构定理)

<u>CLAIM</u>: M_2 contains no closed incompressible surface.

(hence from irreducible + $\partial M_2 \neq \emptyset \Rightarrow$ <u>M_2 consists of handlebodies!!</u>)

<u>Pf</u>: If NOT. $\exists\, \Sigma \subseteq M_2$ is closed and incompressible, then Σ is also closed and incompressible in M_0. Recall S_0 is maximal in M_0, then Σ would be parallel to a component of S_0, and <u>cobound</u> a $\Sigma \times [0,1]$ in M_0, $\Sigma = \Sigma \times \{1\}$.

$S_0^i = \Sigma \times \{0\}$ for some i. $S_0^i \subseteq S_0 \subseteq \boxed{\partial M_1 = \partial M_0 \cup S_0}$

Since S_1 is the spanning surface in M_1, then $\exists\, j$. st. $S_1^j \subseteq S_1$ is one of the components, and $S_1^j \cap S_0^i$ non-trivial.

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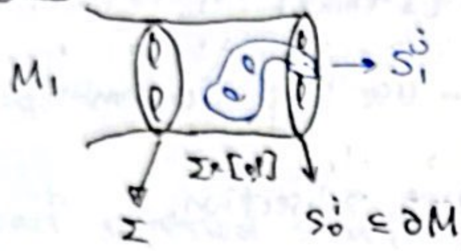
i.e. $[S_1^j \cap S_0^i]$ is nontrivial in $H_1(S_0^i)$.

Note that $\Sigma \subseteq M_2 = M_0 \setminus S_0 \cup S_1$, thus $S_1^j \cap \Sigma = \phi$,

then $\underbrace{S_1^j \subseteq \Sigma \times [0,1]}$ explination



(因为一方面 S_1^j 完全落在 M_1 中，且 S_0^i 为一个边界，因此 S_1^j 必然出现在图上左侧区域。而由 collar lemma，$\Sigma\times[0,1]$ 恰好就是 S_0^i 的邻域。又 $S_1^j \cap \Sigma = \phi$，从而不会穿过 $\Sigma = \Sigma \times \{1\}$，故而一定完全落在 $\Sigma\times[0,1]$ 中)

Hence S_1^j is an (incomp & ∂-incomp) surface in $\Sigma\times[0,1]$, the line bundle. but recall these surfaces only two types:

- horizontal: $\gamma\times[0,1]$, $\gamma \subseteq S_g$ (与 $\Sigma\times[0,1]$ 边界相交，且两个均交)
- vertical: $S_g \times \{\frac{1}{2}\}$. (与边界完全不交)

从而不存在只与单侧边界相交的 (incomp & ∂-incomp) surface! 矛盾!

claim #

Finally, M_2 由 handlebody 组成!

step3 consider essential discs, for each handlebody, then chose S_2 as ($\bigsqcup$ essential discs)

$$\Rightarrow M_2 \overset{S_2}{\rightsquigarrow} M_3 \text{ consist of balls.}$$

$\square$

use hierarchy, we can prove the following:

Thm: $S \subseteq M$ is incompressible $\Leftrightarrow$ $i_* : \pi_1(S) \to \pi_1(M)$ is embedding.

But this proof is technical and (I think) not so much necessary, hence we will use traditional way — use loop theorem to prove it. this details will be methioned in next subsection.

Corollary: Haken mfld has infinite fundamental grp. hence Elliptic 3-mfds are not Haken.

[universal cover S^3 is cmpt. hence finite covering $\Rightarrow \pi_1$ finite.]

[Virtually Haken Conj. (Agol, 2013)
Every irreducible 3-mfld with infinite fundamental grp has a finite covering Haken]

- Notations and Future goals

Topologists sometimes use the term "essential" to summrize various reasonable notions in a single word.

Def: Let M be a cmpt oriented 3-mfld. $S \subseteq M$ is properly embedded connect compact surface, then

- sphere S is called essential, if it NOT bounds a ball.
- disc S is called essential, if it NOT ∂-parallel.
- $\chi(S) \leq 0$ surface S is called essential, if it is
  - (incomp & ∂-incomp) and - NOT ∂-parallel

essential surfaces.

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Def: Let M be irred. & ∂-irred, we say that

- M is atoroidal if it does NOT contain essential tori.
- M is acylindrical if it does NOT contain essential annuli.

Finally, we define:

Def: A compact oriented 3-mfld with (possibly empty) boundary is $\boxed{\text{simple}}$, if it is irred, ∂-irred, atoroidal, acylindrical.
i.e. it does NOT contain essential
spheres, discs, tori, annuli.

Prop: Every closed elliptic / hyperbolic 3-mfld M is simple.

Proof: Since closed, only need to prove irred & atoroidal.

- irred from universal cover (or π_2)
- atoroidal from π_1. i.e. ellipit / hyperbolic mfld's π_1 does not contain $\mathbb{Z}\times\mathbb{Z}$.

$\Downarrow$ π_1 finite

$\Downarrow$ 「四机之间若P含Z×Z 则⇒P不稳数.」

□

Future Goal: We will decompose every irr. and ∂-irred mfld M along some canonical set of essential tori and annuli into some pieces:

- simple mflds;
- Seifert mflds; → 「we will intoduce it and classify it completely.」
- ...

III. Digression: Loop thm and its Application.

Loop Thm: Let M be a 3-mfld with boundary. Suppose there is a map $f: (D^2, \partial D^2) \to (M, \partial M)$ such that $f|_{\partial D^2}$ is NOT null-homotopic in ∂M. Then $\exists$ an embedding $\tilde{f}: (D^2, \partial D^2) \to (M, \partial M)$ with the same property.

i.e. 可以将圆盘映射优化为嵌入.

Corollary: (Dehn's Lemma)

If α is an embedded ~~null-homotopic~~ circle in ∂M, null-homotopic in M then α bounds an embedded disc in M.

Proof: Consider the tubular nbhd of α in ∂M. i.e. an annulus A. then consider $M' = M \setminus (\partial M \setminus A)$. (也即得到个开流形). $\partial M' = A$. then α is NOT null-homotopic in $\partial M'$. Then Apply the loop thm to M' to obtain a disk $D^2 \subseteq M'$ with ∂D^2 non-trivial in $\partial M'$, this boundary isotopic to α. $\square$

Question: ① 曲面上零伦嵌入曲线是否一定 bound disc?

② 若①对. 由 collar lemma 把曲面上 Disk 往里推 不行吗?

Corollary: $S \subseteq M$ properly embedded, cmpt oriented. If the induced map $\pi_1(S) \to \pi_1(M)$ is NOT injective, then $\exists$ disc $D^2 \subseteq M$ with $D^2 \cap S = \partial D^2$ a nontrivial circle in S, i.e. S has a compressing disc. (M里曲线α零伦，则一定存在disc. 考虑α的一个annulus(在S))

Proof: consider $f: D^2 \to M$ be the nullhomotopy of a nontrivial loop in S, then choose $M' = M|S$ apply loop theorem, we can find a disc in M' hence in M. $\square$

Here is another application of the loop thm:

$$\boxed{\text{Prop: A cco prime 3-mfld with } \pi_1(M) \cong \mathbb{Z} \text{ is either } S^1 \times S^2 \text{ or } S^1 \times D^2.}$$

Proof: Case I If $\partial M = \phi$, then from Poincaré duality:

$$\mathbb{Z} \cong H_1(M) \cong H^1(M) \cong H_2(M)$$

Recall $\forall \alpha \in H_2(M)$ can be represented by an (incomp & ∂-incomp) surface, hence in this case, $\exists$ an embedded closed incomp surface S represents $1 \in H_2(M)$.

Since incomp, then each component of S is π_1-injectivity. But $\pi_1(M) \cong \mathbb{Z} \Rightarrow S$ consists of spheres. Recall separating spheres all bound balls from prime $\Rightarrow$ separating spheres are homological trivial.

But $[S]=1 \in H_2(M) \Rightarrow \exists$ Non-separting spheres, then !!

we know that $M = N \# (S^2 \times S^1)$, from prime $\Rightarrow M \cong S^2 \times S^1$.

Case II If $\partial M \neq \phi$, then $M \neq S^2 \times S^1$, hence from M prime we know M is irreducible. then ∂M cannot have sphere, otherwise $M \cong B$. contradicting from $\pi_1(M) \cong \mathbb{Z}$.

From Poincaré-Lefschetz duality:

$\mathbb{Z} \cong H^1_i(M) \cong_{PL} H_2(M, \partial M)$, recall $\partial_{i*}: H_2(M, \partial M) \longrightarrow H_1(\partial M)$ is Lagrangian $\Rightarrow \dim H_1(\partial M) = 2$. $\Rightarrow \partial M$ is a single torus.

But $\pi_1(\partial M) \cong \mathbb{Z} \times \mathbb{Z}$, then $i_*: \pi_1(\partial M) \to \pi_1(M)$ is not injective, then $\exists$ compressing disc. after compession $\rightsquigarrow M'$. $\partial M' = S^2$. M' is also irreducible $\Rightarrow M'$ is ball $\Rightarrow M$ is M' attach 1-handle $\Rightarrow M$ is a solid torus. i.e $S^1 \times \mathbb{D}^2$.

$\square$

Prop: Irreducible cco 3 mfld. $\pi_1(M) = F_g$. then $M \cong H_g$.

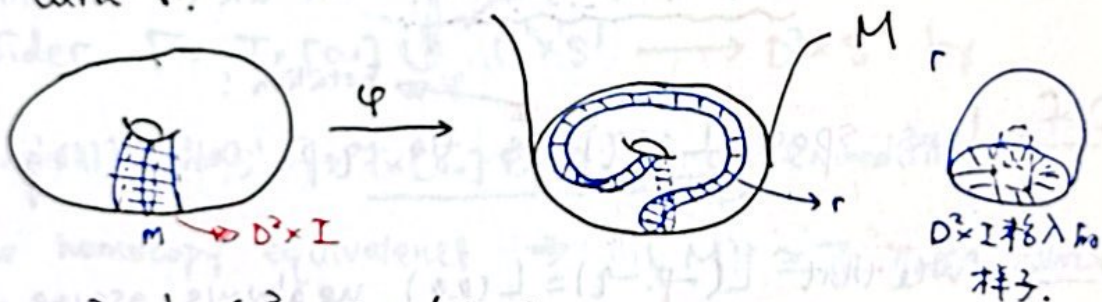
Chapter 10 Seifert M-folds

§ 10.1 Lens spaces

<u>Def</u>: Let M be a 3-mfld and $T \subseteq \partial M$ be a boundary torus component. A <u>Dehn filling</u> of M along T is the operation of gluing a solid torus $D \times S^1$ to M, via a diffeomorphism $\varphi: \partial D \times S^1 \to T$. i.e. $M^{fill} := M \cup_\varphi D \times S^1$

We consider the meridian $m = \partial D \times \{y\}$. and the longitude $l = \{x\} \times S^1$. Suppose m is glued to some simple closed curve <u>$r \subseteq T$</u>, then actually we have:

<u>Prop</u>: The mfld M^{fill} depends only on the isotopy class of the <u>unoriented</u> curve r. → 无行计定向.



<u>Proof</u>: consider $D^2 \times S^1 = (D^2 \times I) \cup (D^2 \times J)$, then firstly glue $D^2 \times I$ to M as attach 2-handle, this is determined by the isotopy class of unoriented r completely.

Then attaching 3-handle (i.e. a ball) does not depends on the attaching map. □

Actually, the Dehn filling opeartion is to use meridian m, and its disc $\mathbb{D}$ to $\underline{\underline{\text{kill}}}$ $r \subseteq T$. More precisely:

Prop: $\pi_1(M^{fill}) \cong \pi_1(M)/N(r)$.

Proof: From SVK.

$$\pi_1(M^{fill}) \cong \pi_1(M \cup D^2\times I) *_{\pi_1(S^2)} \pi_1(D^2\times J)$$

$$\cong \pi_1(M \cup D^2\times I)$$

$$\cong \pi_1(M) *_{\pi_1(\partial D\times I)} \pi_1(D^2\times I) \cong \pi_1(M)/\langle \underbrace{i_* m}_{\text{i.e. the curve } r} \rangle$$

$$\cong \pi_1(M)/N(r).$$

$\square$

Now we note that $\varphi: D\times S^1 \to T$ is in $\mathrm{Mod}(T^2) \cong PSL_2(\mathbb{Z})$, and $\varphi(m.l) := (m.l)\begin{pmatrix} q & r \\ p & s\end{pmatrix}$, $r = \varphi(m) = qm+pl$. we call this is a $(q.p)$ Dehn-filling i.e. use m to kill $qm+pl$.

Def: Lens space $\underline{L(p.q)}$ is the $(q.p)$ Dehn-filling of $M = S^1\times D^2$. notation!

Rmk: Note that $L(-p.-q) = L(p.q)$, we always assume $p \geq 0$.

Exercise: $\pi_1(L(p.q)) \cong \mathbb{Z}/p\mathbb{Z}$.

Proof: $\pi_1(L(p.q)) \cong \mathbb{Z}(l)/\mathbb{Z}(q\overset{i_*m=0}{m}+pl) \cong \mathbb{Z}(l)/\mathbb{Z}(pl) \cong \mathbb{Z}/p\mathbb{Z}$.

$\square$

Prop: $L(0,1) \cong S^2\times S^1$, $L(1,0) \cong S^3$.

Proof: • $L(0,1)$ is to use m to kill $0\cdot l + 1\cdot m = m$. i.e. we have two meridians are attached:

two discs $\longrightarrow \varphi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow L(0,1) \cong S^2\times S^1$.

• $L(1,0)$ is to use m to kill $1\cdot l + 0\cdot m = l$. i.e. attach m to l. and l to m. $\longrightarrow \varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, this is S^3 from $S^3 = \partial(D^2\times D^2) = (D^2\times S^1) \cup (S^1\times D^2)$. □

Exercise: $\forall$ Dehn filling for one component of $T\times[0,1]$ we always get solid torus $D\times S^1$. hence twice Dehn filling for $T\times[0,1]$ we will get lens spaces.

Proof: consider $F: T\times[0,1] \cup_{\varphi} D^2\times S^1 \longrightarrow D^2\times S^1$ by retraction (along $T\times\{1\}$), and $i: D^2\times S^1 \to \underbrace{T\times[0,1]\cup_\varphi D^2\times S^1}_{:=M}$ as embedding.

then F. i are homotopy equivalence $\Rightarrow \pi_1(M) \cong \mathbb{Z}$. $\pi_2(M) = 0$

$\Rightarrow M$ is irreducible with $\pi_1(M) \cong \mathbb{Z}$. recall last prop of last chap

$\Rightarrow M \cong S^1\times D^2$. □

Our Next Goal is to classify all lens space up to diffeomorphism. firstly:

Exercise: If $q' \equiv \pm q^{\pm 1} \pmod p$, then $L(p,q) \cong L(p,q')$.

Actually this is "$\Leftrightarrow$"

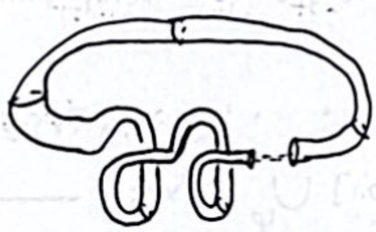
We first state an easy observation:

Observation: If $M_1 \cong M_2$, and $f: M_1 \to M_2$ gives the homeo. then for attaching $M_1 \cup_\varphi N$, we have it is diffeomorphism to $M_2 \cup_{f^{-1}\circ\varphi} N$.

从而为找同胚. 然后找到实现边界同胚的自同胚即可!!

Proof of exercise:

- If $q \equiv q' \pmod p$, then $m \mapsto pl + qm$, $m \mapsto p(l+km) + qm$. Note that we have a self diffeomorphism of $H_1 = D^2 \times S^1$, makes $l \mapsto l + km$. i.e. Dehn twist k times along m.



$(x, e^{i\theta}) \mapsto (x e^{ik\theta}, e^{i\theta})$

- If $q \equiv -q' \pmod p$, then consider the reflection of H_1. $(x, e^{i\theta}) \mapsto (x, e^{-i\theta})$, $m \mapsto -m$.

- Then consider $H_1 \cup_\varphi H_2$ and $H_2 \cup_{\varphi^{-1}} H_1$. (they are diffeo.) If $\varphi = \begin{pmatrix} q & r \\ p & s \end{pmatrix}$,

$\varphi(m,l) = (m,l)\begin{pmatrix} q & r \\ p & s \end{pmatrix}$, $\Rightarrow \varphi^{-1}(m,l) = (m,l) \cancel{\begin{pmatrix} q & -p \\ -r & s \end{pmatrix}} \begin{pmatrix} s & -r \\ -p & q \end{pmatrix}$

$\Rightarrow H_1 \cup_\varphi H_2 \cong L(p,q)$. $H_2 \cup_{\varphi^{-1}} H_2 \cong L(-p,s) \cong L(p,-s) \cong L(p,s)$

Note that $qs - pr = 1 \Rightarrow qs \equiv 1 \bmod p. \Rightarrow s \equiv q^{-1} \pmod p$

$\square$

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§ 10.2 Circle Bundles.

Def: (trivial circle bundle over surface)

Let S be a compact connected surface, consider $S \tilde{\times} I$ to be the unique orientable line bundle over S ($\partial(S \tilde{\times} I)$ = double cover of S).

Then consider the double of $S \tilde{\times} I$ along its boundary ⇒ $S \tilde{\times} S^1$ is an orientable circle bundle over S. Called the trivial one.

This section. our goal is to classify all circle bundles over compact surface:

- If $\partial S \neq \phi$. $\exists !$ orientable circle bundle. $S \tilde{\times} S^1$
- If $\partial S = \phi$. $\{\text{circle bundle over } S\} \cong \mathbb{Z}$ (Euler number)

Rmk: Roughly speaking. If we classify S^1-principal bundle over S

$$\underset{\text{associate bundle.}}{\Longleftrightarrow} \text{ complex line bundle over } S \underset{\text{first chern class}}{\overset{c_1}{\Longleftrightarrow}} H^2(S;\mathbb{Z}) \cong \begin{cases} 0 & \partial S \neq \phi \\ \mathbb{Z} & \partial S = \phi \end{cases}$$

But S^1-fibe bundle "VS" S^1-principal bundle. I have no idea to prove that they are equivalent up to homeomoprhic as manifold.

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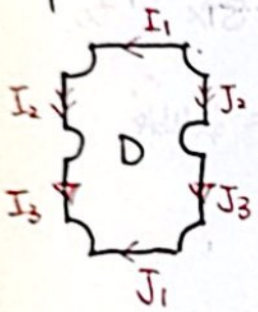
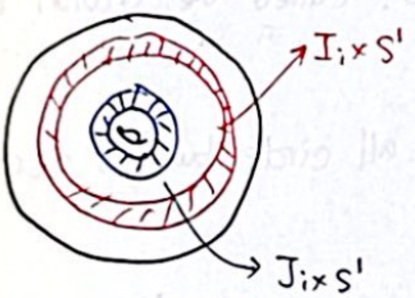
Prop: If $\partial S \neq \phi$. the orientable circle bundles on S all isomophic.

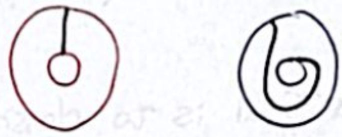
Proof: consider polygon representation (Eg. $S_{1,2}$)



$\Rightarrow$ Circle bundle over S $=$ $D \times S^1$ attach annulus $I_i \times S^1$ and $J_i \times S^1$. Note that all I_i. J_j are disjoint.

Note that this attachment is fibrewise preserving.





$\Rightarrow$ 由黑线决定. Cut along the segment.

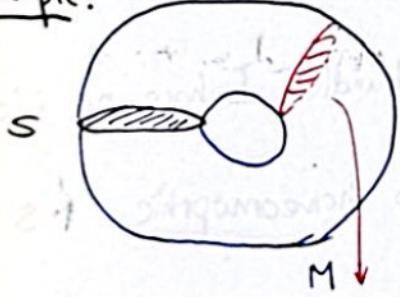
From $\mathrm{Mod}(D^2)$ is trivial $\Rightarrow$ the attachment map isotopic to id

$\Rightarrow$ only have trivial one! $\square$

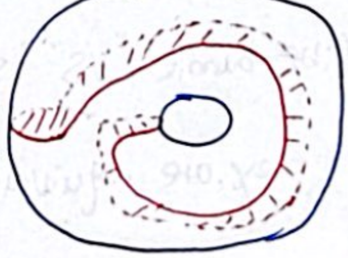
For trivial circle bundle $M \to S$. Now we study the section

$i: S \to M$. s.t. $\pi \circ i = id$.

<u>Example</u>:

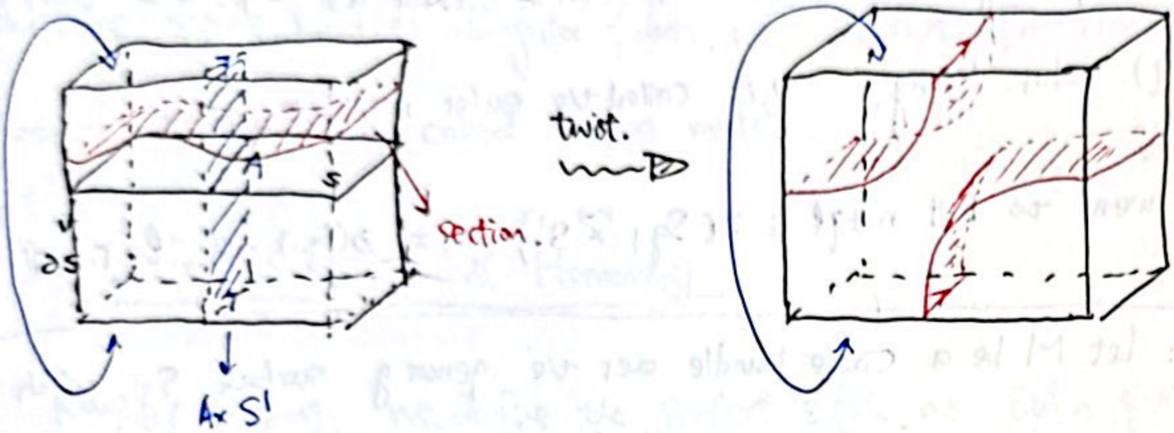


trivial section.



twist section.

We can twist a section along an annulus $A \times S^1$. A is a properly embedded arc in S.

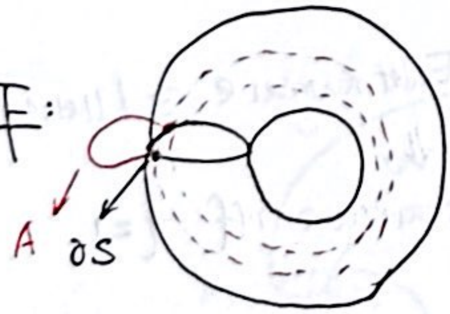


on the boundary. the section of boundary is just the Dehn twist along the circle $(\partial A) \times S^1$.

FACT: Two sections of $S \tilde{\times} S^1$ are connected by a composition of twists along fibered annuli and fibre-preserving isotopies.

Corollary: If S has only one component, the boundary of a section of $M = S \tilde{\times} S^1$ is a slope in ∂M that does not depend on the section.

Proof:



each section can be obtained by twist. then the boundary is positive twist along and negative twist along the other.

$\Rightarrow$ boundary is fixed. $\square$

Now let's classify the closed circle bundles.

FACT: Every S_g circle bundle M can be viewed as $S_{g,1} \tilde{\times} S^1$ then do $(1,q)$ Dehn filling. q is called the euler number.

$\downarrow$

$\Leftarrow$ to kill $m+ql \subseteq \partial(S_{g,1} \tilde{\times} S^1)$ $m = \partial(S_{g,1}) \times \{1\}$. $l = \{1\} \times S^1$.

Exercise: Let M be a circle bundle over the genus g surface S_g with Euler number e, we have $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g} \times \mathbb{Z}/e\mathbb{Z}$.

Solution: $M = (S_{g,1} \times S^1) \cup_{T^2} (D^2 \times S^1)$. From SVK, we have

$$\pi_1(M) = (F_{2g} \times \mathbb{Z}(l)) * \mathbb{Z}(\partial l') \Big/ (\overset{\circ}{m'} = m + el.\ \overset{l'=l}{m = [a_1,b_1]\cdots[a_g,b_g]})$$

$$\Rightarrow H_1(M) = \pi_1(M)^{ab} = (\mathbb{Z}^{2g} \times \mathbb{Z}(l) \times \mathbb{Z}(\partial l')) \Big/ (m' = m + el,\ m = 0,\ l' = l)$$

$$= \mathbb{Z}^{2g} \times \mathbb{Z}/e\mathbb{Z}.$$

$\square$

Corollary: Let $M \to S_g$ and $M' \to S_{g'}$ be circle bundles with Euler number e, e'. then $M \cong M' \Leftrightarrow g = g'$. $|e| = |e'|$.

Exercise: The circle bundle M over S^2 with Euler number $e \cong L(|e|, 1)$.

$\Downarrow$

Corollary: $L(p,q)$ admits S^1 fibre bundle structure iff $q = 1$.

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§ 10.3 Seifert Manifolds.

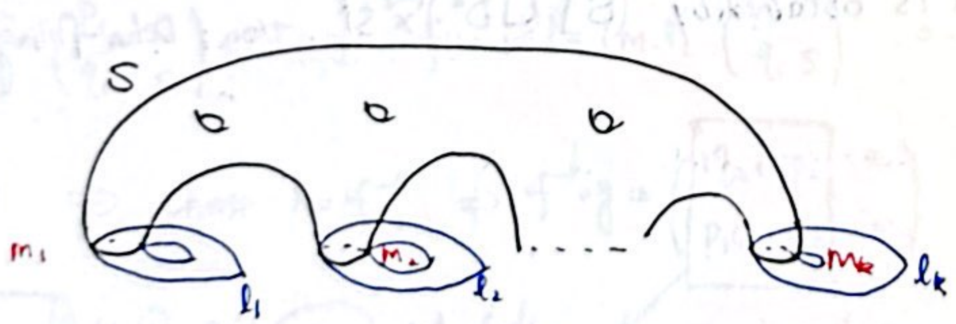
We now enlarge the class of circle bundles over surfaces by admitting some kind of singular fibers, i.e. Seifert fibrations, whose total space are called seifert mflds.

I. Seifert fibrations & Homology.

Roughly speaking, we define the seifert Mflds as Dehn fillings of trivial bundles over surfaces with boundary.

Setting:
- S is a compact connected surface S with $\partial S \neq \emptyset$.
- $M = S \overset{(\sim)}{\times} S^1$ be the unique orientable circle bundle.
- $T_1 \cdots T_k$ be the boundary tori of M.
  - $m_i = T_i \cap \partial S$
  - l_i = the fibre of the bundle.
- A (p_i, q_i) Dehn filling on T_i is to use a solid torus to kill $p_i m_i + q_i l_i$.

We say that the Dehn filling of M is fibe-parallel. if $p_i = 0$.



Def: A seifert mfld is any 3-mfld N obtained from M by Dehn filling the boundary tori in a non-fiber-parallel way. that is $p_i \neq 0$ for all i.

Rmk: If especially we take $k=1$. $p_1=1$, then we get a really circle bundle.

Notation: we use $N=(\hat{S}, (p_1,q_1), \cdots, (p_k,q_k))$ to denote the seifer mfld defined as above. where $\hat{S}$ is the closed surface after capping off S.

Rmk: One may think whether this defn is well, i.e. if we have

$(\{T_1,(p_1,q_1)\} \cdots \{T_k,(p_k,q_k)\})$ VS $(\{T_{\sigma(1)},(p_1,q_1)\}, \cdots \{T_{\sigma(k)},(p_k,q_k)\})$

they actually homeomorphic. ⌜Because every permutation of the boundary circles of S is realized by a self diffeomophism of S, that extends orientation-preservingly to the orientable I-bundle and its double M.⌟

Summary: From now on, we will use $\overset{N}{\overset{/\!/}{(S, (p_1,q_1), \cdots, (p_k,q_k))}}$ to present a seifert mfld. where S is a surface (with possibly empty boundary) then N is obtained by $(S \setminus \overset{k}{\underset{i=1}{\sqcup}} D^2) \tilde{\times} S^1$, then Dehn filling.

Example:

- Seifert mfld $(S_g,(1,e))$ is the circle bundle over S_g with Euler number e. (or the unit bundle of complex line bundle over S_g with $c_1 = e$)

  In particularly $(S_g,(1,0)) = S_g \times S^1$. the trivial one.

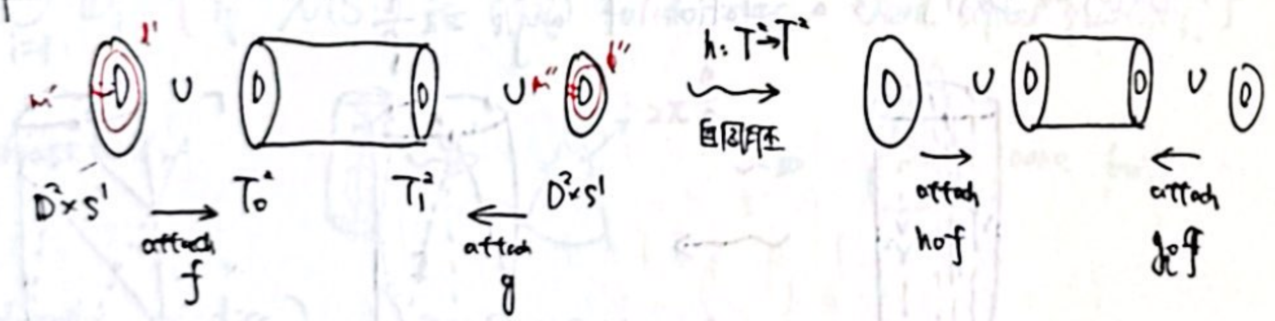
- Seifert mfld $(D^2,(p,q))$ is always solid torus.

  ↳ Dehn filling for $T^2\times[0,1]$.

  ($(T^2\times[0,1]) \cup_f (D^2\times S^1) \cong (T^2\times[0,1]) \cup_{id} (D^2\times S^1)$ by extend f as a automorphism of $T^2\times[0,1]$).

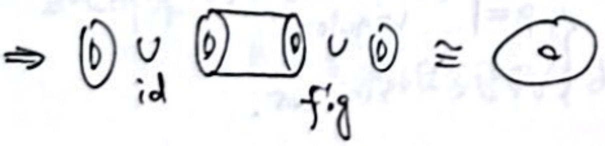
- Seifert mfld $(S^2,(p_1,q_1),(p_2,q_2))$ is the Lens space $L(p_1q_2+q_1p_2,\ rq_2+sp_2)$ where r,s such that $p_1s - q_1r = \pm 1$. In particular, the seifert mfld is S^3 when $p_1q_2+q_1p_2 = \pm 1$.

Proof: Basic philosphy is



$\Rightarrow$ Now for $f = \begin{pmatrix} p_1 & r \\ q_1 & s \end{pmatrix} \in SL(2,\mathbb{Z})$, $f(m',l') = (m,l)\cdot\begin{pmatrix} p_1 & r \\ q_1 & s\end{pmatrix}$. and

$g = \begin{pmatrix} p_2 & x \\ q_2 & y \end{pmatrix} \Rightarrow$ chose $h = f^{-1} \Rightarrow f^{-1}\circ g = \begin{pmatrix} \boxed{\begin{matrix} rq_2+sp_2 \\ p_1q_2+q_1p_2 \end{matrix}} & \begin{matrix} * \\ * \end{matrix} \end{pmatrix}$



Dehn filling $\Rightarrow$ get $L(p_1q_2+q_1p_2,\ rq_2+sp_2)$

$\square$

Goal: Classify all seifert mflds up to homeomorphism. But we need to take it into several steps.

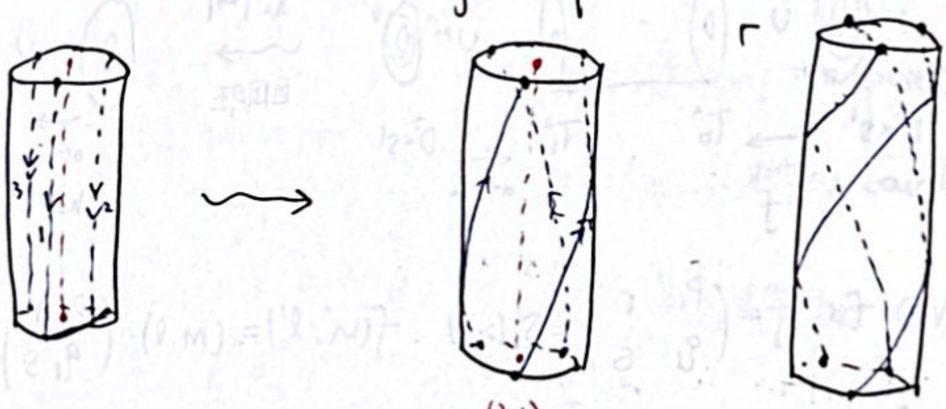
The first step is to view the seifert mflds as a "singular" circle bundle ⇝ siefert fibration. then classify all seifert fibration up to fibration isomophism. (But the problem is that the same mfld may admits different fibration structure. e.g. $L(51,20) = (S^2,(20,51)) = (S^2,(5,3),(7,6))$.).

Roughly speaking, seifert fibration is a partition of circles. and each circle has a "standard fibed solid torus" nbhd:

Def: Let (p,q) be two coprime integers with $p>0$. A standard fibred solid torus with coefficients (p,q) is the solid torus

$$\mathbb{D}\times[0,1]\,/\,\psi$$

$\psi: \mathbb{D}\times\{0\}\to\mathbb{D}\times\{1\}$ is a rotation of angle $2\pi\frac{q}{p}$



(7.1) (1.2)

(把 H_1 分成不同的 S^1- foliation)

$p>0$ is called the multiplicity. $\begin{cases} p=1 & \text{regular} \\ p>1 & \text{singular.}\end{cases}$

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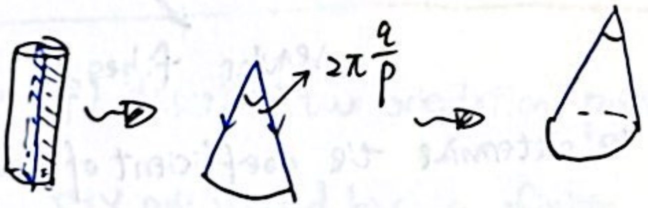
Def: A Seifert fibration is a partion of a compact oriented 3-mfld N with (possibly empty) boundary into circles such that every circle has a fibered nbhd diffeomophic to a standard fibered solid torus.

We wonder what is $\underline{\underline{S}}$, the topological space obtained from N by quotienting circles to points.

Prop: Topologically, S is a cmpt connected surface with (possibly empty) boundary. More precisely, S is an orbifold.

Proof: Note that $N = \bigcup_{i=1}^{k} \mathcal{N}(S_i^1)$, each $\mathcal{N}(S_i^1)$ is a standard solid torus. finite comes from N cpct. $N/S^1 = \bigcup_{i=1}^{k} (\mathcal{N}(S_i^1)/S^1) = \bigcup_{i=1}^{k} D_i^2$ (if $\mathcal{N}(S_i)$ is (p,q) solid torus, then after quotient, we have only



cone but topologically $\cong$ disc) $\Rightarrow S \cong$ surface topologically.

But diff-topologically, S is an orbifold. $\square$

↓ 直观讲来 S 就是一堆标准 disc 与 cone 沿边界粘起来.

(不存在多于两个 disc 共同相粘 ⟶ orbifold)

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$\Rightarrow$ the fibration $N \to S$ is actually a circle bundle over the orbifold S.

A seifert fibration without singular fibres is just an ordinary circle bundle. We now show that seifert mflds and seifert fibrations are more or less the same thing:

Prop: The seifert mfld

$$N = (S, (p_1, q_1), \cdots, (p_n, q_n))$$

has a seifert fibration $N \to S$ over the orbifold

$$(S, p_1, \cdots, p_n).$$

And every seifert fibration arises in this way.

Proof: (mfld → fibration) Easy. $N = \left\{ \left(S \setminus \bigsqcup_{i=1}^{n} D_i^2\right) \tilde{\times} S^1 \right\} \cup$ Dehn fillings

regular fibres   singular fibres.

Now it suffices to determine the coefficient of the singular fibres. suppose we use m to kill $p_i m_i + q_i l_i$. then consider $l = r_i m_i + s_i l_i$ where $p_i s_i - q_i r_i = 1 \Rightarrow l_i = p_i l - r_i m$. hence the coefficients are $(p_i, -r_i) \Rightarrow (S, p_1, \cdots, p_n)$.

(fibration → mfld) remove singular fibre then Dehn filling. $\square$

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We have seen that the notation

$$N = (S, (p_1, q_1) \cdots (p_n, q_n)) \qquad (甲)$$

has two meaning $\begin{cases} \text{seifert mfld } N \\ \text{seifert fibration } N \to (S, p_1, \cdots, p_n). \end{cases}$

$\boxed{\text{Our Next Goal}}$ classify all seifert fibration.

Def: Two seifert fibrations $\pi_1 : N_1 \to S$, $\pi_2 : N_2 \to S$ are isomorphic if $\exists$ a diffeomorphism $\psi : N_1 \to N_2$ such that

$$\begin{tikzcd} N_1 \arrow[rr, "\psi", "\cong"'] \arrow[dr, "\pi_1"'] & \circlearrowleft & N_2 \arrow[dl, "\pi_2"] \\ & S & \end{tikzcd}$$

Two different notations as in (甲) may describe isomorphic fibrations (also diffeomorphic seifert mflds), but this phenomenon is completely understood.

Prop: Two notations in (甲) describe two orientation preservingly isomorphic seifert fibrations iff they are related by a finite sequence of the following moves and their inverse:

(S1): $\cdots (p_i, q_i)(p_{i+1}, q_{i+1}) \cdots \longmapsto \cdots (p_i, q_i + p_i)(p_{i+1}, q_{i+1} - p_{i+1}) \cdots$

(S2): $(p_1, q_1) \cdots (p_n, q_n) \longleftrightarrow (p_1, q_1) \cdots (p_n, q_n)(1, 0)$

(S3): if $\partial N \neq \emptyset$, then $\cdots (p_i, q_i) \cdots \longmapsto \cdots (p_i, q_i + p_i) \cdots$.

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