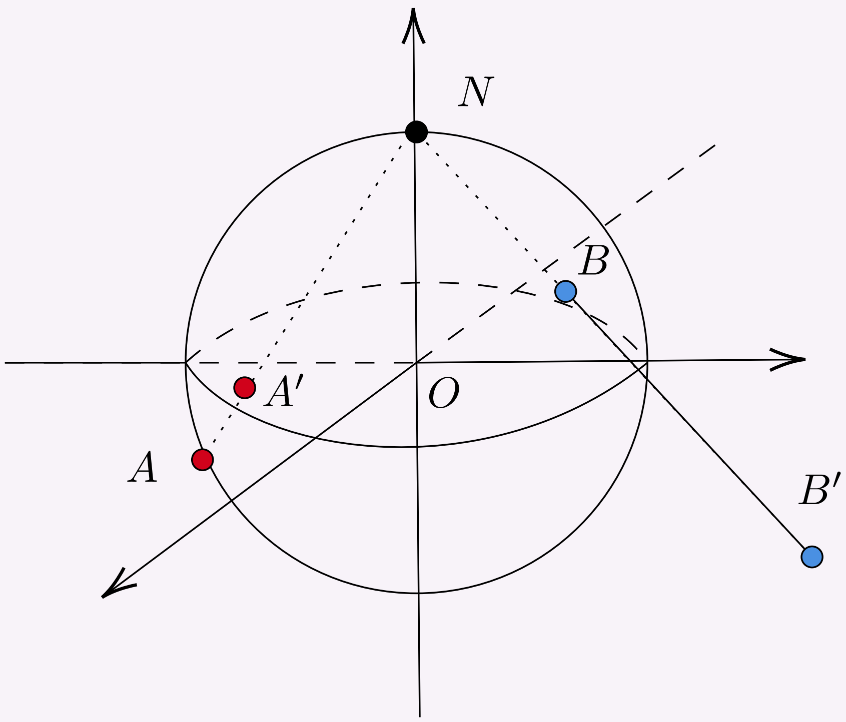
Riemannian Geometry

Notes and Problems



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Preface

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Theorem 1.1.1

We consider

$$\sum_{k=1}^{n}$$

Proof. Firstly ♣

2 Riemannian Metrics

2.1 Notes

Proposition 2.1.1: Polarization Identity

Suppose $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V, then

$$\langle v, w \rangle = \frac{1}{4}\left(|v+w|^2 - |v-w|^2\right).$$

Proof. (**Exercise 2.2**) Using bilinearity and expand $|v \pm w|^2$. ♣

Definition 2.1.1: Riemannian Metrics

Let M be a smmoth manifold, a **Riemannian metric** on M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M) = \Gamma(T^k T^* M)$, whose value g_p at each $p \in M$ is an inner product on $T_p(M)$, and for all $X, Y \in \mathfrak{X}(M)$, $g(X,Y)(p) := g_p(X_p, Y_p)$ is a smooth function on M.

Proposition 2.1.2

Every smooth manifold admits a Riemannian metric.

Proof. (**Exercise 2.5**) Choose an atlas $\{\varphi_\alpha | U_\alpha \to V_\alpha \subseteq \mathbb{R}^n\}$, and a subordinate partition of unity $\{\rho_\alpha\}$, suppose g_α is Euclidean inner product in V_α, then for each $X, Y \in \mathfrak{X}(M)$, we can define

$$g(X,Y) = \sum_\alpha \rho_\alpha g_\alpha((\varphi_\alpha)_* X, (\varphi_\alpha)_* Y),$$

it is trivial to check the definition above is as desired. ♣

Definition 2.1.2: Isometry and Local Isometry

Suppose (M, g) and (M', g') are Riemannian manidfols, an **isometry** from (M, g) to (M', g') is a diffeomorphism $\varphi : M \to M'$ such that $\varphi^* g' = g$.

A map $\varphi : M \to M'$ is a **local isometry** if each point $p \in M$ has a neighborhood U such that $\varphi|_U$ is an isometry onto an open subset of M'.

Remark. Equivalently, $\varphi^* g' = g$ denotes that the differential φ_* at each p is a linear isometry from $T_p(M)$ to $T_{\varphi(p)}(M)$, since $g(X,Y) = \varphi^* g'(X,Y) = g'(\varphi_* X, \varphi_* Y)$.

And note that if φ is a local isometry, we also have $\boxed{\varphi^* g' = g}$, since it is a local result, so the main difference between local isometry and isometry is "map" and "diffeomorphism".

Example 2.1.1 (The isometry of $(\mathbb{R}^n, \overline{g})$)**.** *Let φ be an isometry on $(\mathbb{R}^n, \overline{g})$, then there exists $A \in O(n)$ such that $\varphi(x) = Ax + \varphi(0)$, i.e. we have* $\mathrm{Isom}(\mathbb{R}^n) = O(n) \rtimes \mathbb{R}^n$.

Proof. **(Using Geodesic)** Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, and WLOG we assume $\varphi(0) = 0$, and suppose $\psi = \varphi \circ (\varphi_{*,0})^{-1}$, note that $\varphi_{*,0}$ is the Jacobian of φ at 0, and thus $\psi(0) = 0$ and $\psi_{*,0} = \mathrm{Id}$.

Recall that the isometry sends geodesic to geodesic, then for geodesic $\{tv | t \in \mathbb{R}\}$ with velocity v, then $\psi(tv)$ is also a geodesic, since $\psi(0) = 0$ and $\psi_{*,0}(v) = v$, thus using the uniqueness of geodesic, we know that actually $\psi(tv) = tv$, then $\psi = \mathrm{Id}$, thus φ is a linear constant map, furthermore, it is an orthnogonal matrix. ♣

Remark. A more elementary proof can be found in *Differential Geometry*, Jiagui Peng.

Definition 2.1.3: Flat

A Riemannian $n-$manifold is said to be **flat**, if it is locally isometric to a Euclidean space, that is, if every point has a neighborhood that is isometric to an open set in $\mathbb{R}^n$ with its Euclidean metric $\overline{g}$.

Proposition 2.1.3

Suppose (M', g') is a Riemannian manifold, and F is a smmoth map, then the smooth $2-$tensor field $g = F^* g'$ is a Riemannian metric on M if and only if F is an immersion.

Proof. **(Exercise 2.12)** Recall that F is an immersion iff F_* is injective, thus on the one hand, if $g = F^* g'$ is a metric then suppose $F_{*,p} X_p = 0$, then since $g(X_p, X_p) = F^* g'(X_p, X_p) = g'(F_{*,p} X_p, F_{*,p} X_p) = 0$, then $X_p = 0$ thus F_* is injective then F is immersion.

On the other hand ,suppose F_* is injective then g is naturally non-negative symmetry, the only thing needs to check is $g(X, X) = 0$ iff $X = 0$, and injectivity provides that. ♣

Example 2.1.2 (Standard Metric on Sphere)**.** *The Riemannian metric induced on S^n by the Euclidean metric of $\mathbb{R}^{n+1}$ is denoted by $\mathring{g}$, we will give its precise expansion using local coordinate.*

Metrics of Riemannian Submanifolds

Computationson a submanifold $M^m \subseteq \mathbb{R}^n$ are usually carried out most conveniently in terms of a **smooth local parametrization**: there is a smooth map $X : U \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^m$ such that $X(U)$ is an open subset of $M \subseteq \mathbb{R}^n$.

Then wen put $V = X(U) \subseteq M$ and $\varphi = X^{-1} : V \to U \subseteq \mathbb{R}^m$, then (V, φ) is a smooth coordinate chart on M, then if g is the Riemannian metric on M, and $\iota : M \subseteq \mathbb{R}^n$, the coordinate representation of g in (V, φ) is

$$(\varphi^{-1})^* g = X^* g = X^*(\iota^* \overline{g}) = (\iota \circ X)^* \overline{g} = X^* \overline{g},$$

Thus once we have $X:(u_1,\cdots,u_m)\mapsto(X_1,\cdots,X_n)$, then we ca locally give the coorcinate representation of $X(U)$ in M:

$$g=X^*\overline{g}=\sum_{i=1}^m(\mathrm{d}X^i)^2=\sum_{i=1}^m\left(\frac{\partial X^i}{\partial u^j}\mathrm{d}u^j\right)^2=\sum_{i=1}^m\frac{\partial X^i}{\partial u^j}\frac{\partial X^i}{\partial u^k}\mathrm{d}u^j\mathrm{d}u^k.$$

Example 2.1.3 (Metric of S^n)**.** *Consider the upper semisphere and $X:B^n\to S^n\subseteq\mathbb{R}^{n+1}$:*

$$X(u^1,\cdots,u^n)=\left(u^1,\cdots,u^n,\sqrt{1-(u^1)^2-\cdots-(u^n)^2}\right),$$

we see that the round metric on S^n can be written locally as

$$\begin{aligned}\mathring{g}&=(\mathrm{d}u^1)^2+\cdots+(\mathrm{d}u^n)^2+\left(\mathrm{d}\sqrt{1-(u^1)^2-\cdots-(u^n)^2}\right)^2\\&=(\mathrm{d}u^1)^2+\cdots+(\mathrm{d}u^n)^2+\left(\frac{u^1\mathrm{d}u^1+\cdots+u^n\mathrm{d}u^n}{\sqrt{1-(u^1)^2-\cdots-(u^n)^2}}\right)^2\\&=\boxed{\frac{(1-(u^i)^2)(\mathrm{d}u^i)^2+2u^iu^j\mathrm{d}u^i\mathrm{d}u^j}{1-(u^1)^2-\cdots-(u^n)^2}}.\end{aligned}$$

Example 2.1.4 (Flat Tours)**.** *The $n-$tours $T^n=S^1\times\cdots\times S^1$, regarded as a subset of $\mathbb{R}^{2n}$ by*

$$(x^1)^2+(x^2)^2=\cdots=(x^{2n-1})^2+(x^{2n})^2=1,$$

and the smooth map $X:\mathbb{R}^n\to T^n$ given by

$$X(u^1,\cdots,u^n)=(\cos u^1,\sin u^1,\cdots,\cos u^n,\sin u^n)$$

induces a flat metric on T^n, since

$$g=X^*\overline{g}=\sum_{i=1}^n((-\sin u^i)^2+(\cos u^i)^2)(\mathrm{d}u^i)^2=\sum_{i=1}^m(\mathrm{d}u^i)^2.$$

Riemmanian Products and Warped Products

Definition 2.1.4

Suppose (M_1,g_1) and (M_2,g_2) are two Riemannian manifolds, and $f:M_1\to\mathbb{R}^+$ is a strictly positive smooth function. The **warped product** $M_1\times_f M_2$ is the product manifold $M_1\times M_2$ endowed with the Riemannian metric $g=g_1\oplus f^2g_2$, defined by

$$g_{(p_1,p_2)}((v_1,v_2),(w_1,w_2)):=g_1|_{p_1}(v_1,w_1)+f(p_1)^2\cdot g_2|_{p_2}(v_2,w_2).$$

Remark. This is an improtant concept in *Metric Geometry*. And some interesting examples about warped product showed in the problems.

Example 2.1.5 (Fubini-Study metric)**.** *Note that there is a projection $\pi:S^{2n+1}\to\mathbb{CP}^n$, and it is not hard to show that this is a submersion, then there exists a unique metric g such that π is a Riemannian submersion, i.e., $\mathring{g}(X,Y)=g(\pi_*X,\pi_*,Y)$.*

Basic Constructions on Riemannian Manifolds

We define a bundle homorphism $\hat{g}: TM \to T^*M$ by setting

$$\hat{g}: X \mapsto \hat{g}(X), \quad \hat{g}(X)(Y) = g(X, Y).$$

Suppose $\{E_i\}$ is a smooth local frame, and its dual coframe is (ε^i), let $g = g_{ij}\varepsilon^i\varepsilon^j$, and $X = X^i E_i$, then suppose $\hat{g}(X) = a_i\varepsilon^i$, then we have $a_i = \hat{g}(X)(E_i) = g(X, E_i) = g_{ji}X^j$, thus

$$\hat{g}(X) = (g_{ij}X^i)\varepsilon^i := X_i\varepsilon^i, \quad X_i = g_{ij}X^j.$$

So we say that $\hat{g}(X)$ is obtained from X by **lowering an index**, since the coordinate turns from X^i to X_i, and we denoted $\hat{g}(X)$ by $X^\flat$ and called X **flat**.

Similarly, for a 1-form ω, one can define $\omega^\sharp := \hat{g}^{-1}(\omega)$, in short, we have

$$X^\flat(Y) = g(X, Y), \quad g(\omega^\sharp, Y) = \omega(Y).$$

For $f : M \to \mathbb{R}$ be a smooth function, then grad$f := (\mathrm{d}f)^\sharp$, since $\mathrm{d}f = a_i\varepsilon^i$, then $a_i = \mathrm{d}f(E_i) = E_i f$, then $\mathrm{d}f = (E_i f)\varepsilon^i$, and suppose grad$f = a^i E_i$, then from $a^j g_{ij} = g(\mathrm{grad} f, E_i) = \mathrm{d}f(E_i) = E_i f$, thus $a^j = g^{ij}(E_i f)$.

We can generalize this notation to any tensor, i.e, for

$$A = A_i{}^j{}_k \varepsilon^i \otimes E_j \otimes \varepsilon_k,$$

then we have $A^\flat = A_{ijk}\varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k$, where $A_{ijk} = g_{jl}A_i{}^l{}_k$.

Now from the trace of $(k+1, l+1)$ tensor

$$F = F^{i_1 \cdots i_{k+1}}_{j_1 \cdots j_{l+1}} E_{i_1} \otimes \cdots \otimes E_{i_{k+1}} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_{l+1}},$$

then we define its trace tr$F$ is a (k, l) tensor such that

$$(\mathrm{tr}F)^{i_1 \cdots i_k}_{j_1 \cdots j_l} = F^{i_1 \cdots i_k m}_{j_1 \cdots j_l m} E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}.$$

Now for any $(0, k)$ tensor field h, and

$$h = h_{j_1 \cdots j_k}\varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_k},$$

we define the **trace of h with respect to g** as $\mathrm{tr}_g h = \mathrm{tr} h^\sharp$ is a $(0, k-2)$ tensor, where

$$h^\sharp = h_{j_1 \cdots j_{k-1}}{}^{j_k}\varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_{k-1}} \otimes E_{j_k},$$

where from preceeding discussion, we have

$$h_{j_1 \cdots j_{k-1}}{}^{j_k} = g^{l j_k} h_{j_1 \cdots j_{k-1} l},$$

thus we actually have

$$\mathrm{tr}_g h = h_{j_1 \cdots j_{k-2} m}{}^m \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_{k-2}} = g^{lm} h_{j_1 \cdots j_{k-2} m l}\varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_{k-2}}$$

Inner Products of Tensors

Firstly, we define the inner product between 1-form:

$$\langle \omega, \eta \rangle = \langle \omega^\sharp, \eta^\sharp \rangle,$$

suppose $\omega = \omega_i \varepsilon^i$, then we have $\omega^\sharp = \omega_i E_i = g^{ij}\omega_j E_i$, then for $\eta^\sharp = g^{ij}\eta_j E_i$, so

$$\langle \omega, \eta \rangle = \langle g^{ij}\omega_i E_j, g^{kl}\eta_k E_l \rangle = g^{ij}g^{kl}g_{jl} \cdot \omega_i \eta_k = g^{ik}\omega_i \eta_k.$$

So it is really a natural generalization of inner prodcut of vectors, since

$$g(X, Y) = g_{ij}X^iY^j, \quad g(\omega, \eta) = g^{ij}\omega_i\eta_j.$$

Now we can define the inner products of arbitary (k, l) tensor, i.e., we define

$$\langle \alpha_1 \otimes \cdots \otimes \alpha_{k+1}, \beta_1 \otimes \cdots \otimes \beta_{k+l} \rangle := \langle \alpha_1, \beta_1 \rangle \cdots \langle \alpha_{k+1}, \beta_{k+l} \rangle.$$

And for $F = F^{i_1 \cdots i_k}_{j_1 \cdots j_l} E \otimes \varepsilon$, $G = G^{s_1 \cdots s_k}_{t_1 \cdots t_l} E \otimes \varepsilon$, we now have

$$\langle F, G \rangle = F^{i_1 \cdots i_k}_{j_1 \cdots j_l} \cdot G^{s_1 \cdots s_k}_{t_1 \cdots t_l} \cdot g_{i_1 s_1} \cdots g_{i_k s_k} g^{j_1 t_1} \cdots g^{j_l t_l}.$$

Some Basic Operators abou Differential Forms

If $\{\varphi^i\}$ is any local oriented orthonormal coframe for T^*M, then we define

$$\mathrm{d}V_g = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n.$$

If (η^i) is another orthonormal basis, and $\eta^i = a^i_j \varepsilon^j$ and $a^i_j a^k_j = \delta^i_k$, i.e., (a^i_j) is an orthonormal matrix, then we have $\eta^1 \wedge \cdots \wedge \eta^n = (a^1_i \varepsilon^i) \wedge \cdots \wedge (a^n_i \varepsilon^i) = \mathrm{d}V_g$, so the volume form is not dependent on the choice of orthonarmal basis.

Now more precisely, suppose $\varepsilon^i = b^i_k \mathrm{d}x^k$, then suppose $B = (b^i_k)$, then since $g_{ij} = \langle \partial_i, \partial_j \rangle$, then we have $g(\mathrm{d}x^i, \mathrm{d}x^j) = g^{ij}$, then we have $I_n = BB^T(g^{ij})$, thus, we have $\mathrm{det}B = \sqrt{\mathrm{det}(g_{ij})}$, so we use the similar way to calculate, and have

$$\mathrm{d}V_g = \sqrt{\mathrm{det}(g_{ij})}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

Now we recall some basic operators,

1. **(Exterior Differential)** $\boxed{\mathrm{d} : \bigwedge^*(M) \to \bigwedge^{*+1}(M)}$ and we have for $\omega \in \bigwedge^k(M)$, then

$$\begin{aligned}\mathrm{d}\omega(X_0, \cdots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \cdots, \widehat{X_i}, \cdots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0 \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_k).\end{aligned}$$

2. **(Interior Product)** $\boxed{i(X) : \bigwedge^*(M) \to \bigwedge^{*-1}(M)}$ and then we have for $\omega \in \bigwedge^k(M)$, then

$$i(X)\omega(X_1, \cdots, X_{k-1}) = \omega(X, X_1, \cdots, X_{k-1}).$$

3. **(Lie Derivative)** $\boxed{\mathcal{L}_X : \bigwedge^*(M) \to \bigwedge^*(m)}$, then we have for $\omega \in \bigwedge^k(M)$, then

$$\mathcal{L}_X\omega(X_1, \cdots, X_k) = X(\omega(X_1, \cdots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \cdots, [X, X_i], \cdots, X_k).$$

4. **(Hodge Star)** $\boxed{* : \bigwedge^k(M) \to \bigwedge^{n-k}(M)}$, recall there always have

$$\omega \wedge (*\omega) = \mathrm{d}V_g, \quad \forall \omega \in \wedge^k(M).$$

5. (**Divergence**) $\boxed{\mathrm{div} : \mathfrak{X}(M) \to \mathbb{R}}$, and more precisely,

$$\mathrm{div}(X) := *\mathrm{d} * X^\flat,$$

we also define the divergence as

$$(\mathrm{div}X) \cdot \mathrm{d}V_g := \mathrm{d}(i(X)\mathrm{d}V_g),$$

we will use the second definition in our latter discussion.

6. **(Laplacian Operator)** $\Delta : C^\infty(M) \to C^\infty(M)$, we have

$$\Delta(u) = \mathrm{div}(\mathrm{grad}u),$$

we can actually define $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ for differential forms, but we will not use it until we discuss about Hodge theory.

Here are some relations about the operators above, we will select some of them to prove:

1. $i(X)(\omega \wedge \eta) = i(X)\omega \wedge \eta + (-1)^{\mathrm{deg}\omega}\omega \wedge i(X)\eta;$

2. $\mathcal{L}_X i(Y) - i(Y)\mathcal{L}_X = i([X, Y]);$

3. $\mathcal{L}_X = i(X)\mathrm{d} + \mathrm{d}i(X);$

4. $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X\omega \wedge \eta + \omega \wedge \mathcal{L}_X\eta$, one can view it as Leibniz rule;

5. $\mathcal{L}_X\mathrm{d} = \mathrm{d}\mathcal{L}_X;$

6. $\mathcal{L}X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]};$

Proposition 2.1.4

Let (M, g) be a smooth Riemmannian manifold, and (x^i) ia s local coordinate, then we have

$$\begin{aligned} \operatorname{div}(X^i\partial_i) &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(X^i\sqrt{G}\right) \\ \Delta u &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{G}\frac{\partial u}{\partial x^j}\right). \end{aligned}$$

Proof. Suppose $X = X^i\partial_i$, then since $\mathrm{d}(i(X)\mathrm{d}V_g) = \operatorname{div}(X)\mathrm{d}V_g$, thus we have

$$\operatorname{div}X \cdot \sqrt{G} = \mathrm{d}(i(X)\mathrm{d}V_g)(\partial_1, \cdots, \partial_n).$$

Suppose $i(X)\mathrm{d}V_g = a_i \mathrm{d}x^1 \wedge \cdots \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$, then we have

$$\begin{aligned} a_i &= i(X)\mathrm{d}V_g(\partial_1, \cdots, \widehat{\partial_i}, \cdots, \partial_n) \\ &= \mathrm{d}V_g(X, \partial_1, \cdots, \widehat{\partial_i}, \cdots, \partial_n) \\ &= \sqrt{G}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(X^i\partial_i, \partial_1, \cdots, \widehat{\partial_i}, \cdots, \partial_n) \\ &= \sqrt{G}\cdot(-1)^{i-1}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(\partial_1, \cdots, X^i\partial_i, \cdots, \partial_n) \\ &= \sqrt{G}\cdot X^i \cdot (-1)^{i-1}. \end{aligned}$$

Thus it is trivial to see that we really have

$$\boxed{\operatorname{div}(X^i\partial_i) = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(X^i\sqrt{G}\right)}.$$

Now for $\Delta u = \operatorname{div}(\operatorname{grad}u)$, and since $\operatorname{grad}u = (\mathrm{d}u)^\sharp = u^i\partial_i = \left(g^{ij}\frac{\partial u}{\partial x^j}\right)\partial_i$, since

$$\mathrm{d}u = \frac{\partial u}{\partial x^j}\mathrm{d}x^j := u_j \mathrm{d}x^j,$$

Thus the formula follows from the divergence. ♣

Theorem 2.1.1

Suppose (M, g) is compact oriented manifold, then the folowing divergence theorem holds for $X \in \mathfrak{X}(M)$:

$$\int_M \operatorname{div}X \mathrm{d}V_g = \int_{\partial M}\langle X, N\rangle \mathrm{d}V_{\widehat{g}},$$

where N is the outward unit normal to ∂M and $\widehat{g}$ is the induced metric on ∂M.

Proof. For any p on ∂M, we suppose $N \cup \{E_i\}_{i=2}^n$ is an orthonormal basis of T_pM, and $\{E_i\}_{i=2}^n$ is an orthonormal basis of $T_p\partial M$, and the dual basis is $\varepsilon^1, \varepsilon^2, \cdots \varepsilon^n$, then $\mathrm{d}V_g = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$, and $\mathrm{d}V_{\hat{g}} = \varepsilon^2 \wedge \cdots \wedge \varepsilon^n$ and assume $i(X)\mathrm{d}V_g = a_i\varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^i} \wedge \cdots \wedge \varepsilon^n$, then we have

$$a_1 = \mathrm{d}V_g(X, E_2, \cdots, E_n) = \langle X, N\rangle,$$

suppose $\iota : \partial M \hookrightarrow M$, then by Stokes furmula

$$\begin{aligned}\int_M \mathrm{div}X \mathrm{d}V_g &= \int_M \mathrm{d}(i(X)\mathrm{d}V_g) \\ &= \int_{\partial M} \iota^*(i(X)\mathrm{d}V_g) \\ &= \int_{\partial M} a_1\varepsilon^2 \wedge \cdots \wedge \varepsilon^n \\ &= \int_{\partial M} \langle X, N\rangle \mathrm{d}V_{\widehat{g}}.\end{aligned}$$

Finally ,we complete the proof, the main part is ι^*. ♣

Corollary 2.1.1: Integraton by Parts

The divergence operator satisfies the following product rule

$$\mathrm{div}(uX) = u\mathrm{div}X + \langle \mathrm{grad}u, X\rangle, \quad u \in C^\infty(M), X \in \mathfrak{X}(M),$$

then we have the following integration by parts formula

$$\int_M \langle \mathrm{grad}u, X\rangle \mathrm{d}V_g = \int_{\partial M} u\langle X, N\rangle \mathrm{d}V_{\widehat{g}} - \int_M u\mathrm{div}X\mathrm{d}V_g.$$

Proof. We have

$$\begin{aligned}\mathrm{div}(uX) &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(uX^i\sqrt{G}) \\ &= u\cdot\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(X^i\sqrt{G}) + X^i\cdot\frac{\partial u}{\partial x^i} \\ &= u\mathrm{div}X + Xu = u\mathrm{div}X + \langle \mathrm{grad}u, X\rangle.\end{aligned}$$

And using divergence theorem we can finish the proof. ♣

Theorem 2.1.2: Dirichlet Principle

Suppose (M, g) is a compact connected Riemannian manifold with nonempty boundary. Then a function $u \in C^\infty(M)$ is harmonic if and only if

$$\int_M |\mathrm{grad}u|^2 \mathrm{d}V_g = \min_{f|_{\partial M} = u|_{\partial M}} \int_M |\mathrm{grad}f|^2 \mathrm{d}V_g.$$

Proof. For any fixed $f \in C^\infty(M)$ and $f|_{\partial M} = 0$, then from integration by parts,

$$\int_M u\mathrm{div}X\mathrm{d}V_g + \int_M \langle \mathrm{grad}u, X\rangle \mathrm{d}V_g = \int_{\partial M} u\langle X, N\rangle \mathrm{d}V_{\widehat{g}},$$

then we actually have

$$\int_M af\Delta u \mathrm{d}V_g + \int_M \langle \mathrm{grad}u, \mathrm{grad}af\rangle \mathrm{d}V_g = \int_{\partial M} afNu\mathrm{d}V_{\widehat{g}},$$

Then we have for all $f|_{\partial M}=0$,

$$\begin{aligned}&\int_M |\mathrm{grad}(u+af)|^2\mathrm{d}V_g-\int_M|\mathrm{grad}u|^2\mathrm{d}V_g\\ &=a^2\int_M|\mathrm{grad}f|^2\mathrm{d}V_g+2a\int_M\langle\mathrm{grad}u,\mathrm{grad}f\rangle\mathrm{d}V_g\\ &=a^2\int_M|\mathrm{grad}f|^2\mathrm{d}V_g-2a\int_M f\Delta u\mathrm{d}V_g,\end{aligned}$$

thus on the one hand, if u is harmonic, then for arbitary $f|_{\partial M}=0$, we have

$$\int_M |\mathrm{grad}(u+af)|^2\mathrm{d}V_g-\int_M|\mathrm{grad}u|^2\mathrm{d}V_g=a^2\int_M|\mathrm{grad}f|^2\mathrm{d}V_g\geq 0,$$

i.e., which minimizes the integral. On the onther hand, for arbitary $p\in M^\circ$, (if $M^\circ=\varnothing$, then $M=\partial M$, there is nothing needs to show), then there is $r>0$ such that $B_r(p)\subseteq M$ and consider a smooth cut off function $0\leq\rho\leq 1$ such that it is supported in $B_{r/2}(p)$.

Then consider smooth function $\boxed{g=\rho\Delta u}$ which is naturally vanishes on ∂M, if grad$g$ already identically zero on M, then choose $a>0$ one can see that we muast have

$$K:=\int_M\rho(\Delta u)^2\mathrm{d}V_g=\int_{B_r(p)}\rho(\Delta u)^2=0,$$

if not, then since $K>0$, so we can choose

$$0<a<\frac{K}{\int_M|\mathrm{grad}g|^2\mathrm{d}V_g}\quad\Rightarrow\quad\int_M|\mathrm{grad}(u+ag)|^2\mathrm{d}V_g<\int_M|\mathrm{grad}u|^2\mathrm{d}V_g$$

which is a contradiction, then we know that we must have $K=0$, i.e., $\Delta u\equiv 0$ on $B_r(p)$, now since M is compact, we know $\Delta u\equiv 0$, thus we know that u is harmonic function. ♣

2.2 Problems

Problem 1. Show that every Riemannian 1-manifold is flat.

Proof. Recall that it is suffices to show that every point has a neighborhhood that is isometric to an open set in $\mathbb{R}$ with its Euclidean metric $\overline{g}$. For given $p \in M$, and choose a coordinate neighborhood U of p with coordinate x, then suppose $N(q) = g_q(\partial/\partial x|_q, \partial/\partial x|_q)$ is a smooth function on U, then consider $y = x/\sqrt{N}$, thus $g_q(\partial/\partial y, \partial/\partial y) = 1$, thus (U, y) is as desired. ♣

Remark. Actually, we will soon know that flat means that the curvature vanishes, since the curvature is a 2 -form, it vanishes on a 1-manifold.

Problem 2. Suppose V and W are finite-dimensional real inner product spaces of the same dimension, and $F : V \to W$ is any map satisfies $F(0) = 0$ and $|F(x) - F(y)| = |x - y|$ for all $x, y \in V$, Prove that F is a linear isometry.

Proof. Recall that a linear isometry is a vector space isomorphism which preserves inner product, let $y = 0$ then the latter is trivial, and F is naturally injective, so the only thing needs to prove is that F is linear. Note that $\langle F(av) - aF(v), F(av) - aF(v)\rangle = |F(av)|^2 - 2a^2\langle F(av), F(v)\rangle + a^2|F(v)|^2 = 0$, thus $F(av) = aF(v)$, using the same method one can shows that $F(av + bw) = aF(v) + bF(w)$, then F is linear, then we finish the proof. ♣

Problem 3. Given a smooth embedded 1-dimensional submanifold $C \subseteq H$, then the surface of revolution $S_C \subseteq \mathbb{R}^3$ with its induced metric is isometric to the warped produt $C \times_a S^1$.

Proof. Suppose $C = (a(s), b(s))$ where s is the length paramter, then $a'(s)^2 + b'(s)^2 = 1$, then we have $X(s, \theta) = (a(s)\cos\theta, a(s)\sin\theta, b((s)))$, then g represents by (s, θ) is

$$g = X^*\overline{g} = (a'(s)^2 + b'(s)^2)(\mathrm{d}s)^2 + a^2(s)\mathrm{d}\theta^2 = \mathrm{d}s^2 + a^2(s)\mathrm{d}\theta^2,$$

since it has same formula with $C \times_a S^1$, then the isomorphism is trivial. ♣

Problem 4. Let $\rho : \mathbb{R}^+ \to \mathbb{R}$ be the restriction of the standard coordinate function, and let $\mathbb{R}^+ \times_\rho S^{n-1}$ be the warped product. Define $\Phi : \mathbb{R}^+ \times_\rho S^{n-1} \to \mathbb{R}^n - \{0\}$ by $\Phi(\rho, \omega) = \rho\omega$. Show that Φ is an isometry between them.

Proof. Recall an isometry is diffeomorphism + $\varphi^* g' = g$, since diffeomorphism is trivial, so it suffices to show the metric equation. Since $g = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\omega^2$, and for $\omega = (\omega^1, \cdots, \omega^{n-1}, \sqrt{1-(\omega^i)^2})$, and $\mathrm{d}\omega^2$ is the round metric of S^{n-1},and recall then

$$\begin{aligned}
\Phi^* \overline{g} &= \Phi^* \left((\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2\right) = (\mathrm{d}\Phi^1)^2 + \cdots + (\mathrm{d}\Phi^n)^2 \\
&= \sum_{i=1}^{n} \left(\frac{\partial \Phi^i}{\partial \rho}\mathrm{d}\rho + \frac{\partial \Phi^i}{\partial \omega^j}\mathrm{d}\omega^j\right)^2 = \sum_{i=1}^{n-1} (\omega^i \mathrm{d}\rho + \rho \mathrm{d}\omega^i)^2 + \left(\sqrt{1-(\omega^i)^2}\mathrm{d}\rho + \rho \mathrm{d}\sqrt{1-(\omega^i)^2}\right)^2 \\
&= \mathrm{d}\rho^2 + \rho^2 \left((\mathrm{d}\omega^1)^2 + \cdots + (\mathrm{d}\omega^{n-1})^2 + \left(\mathrm{d}\sqrt{1-(\omega^1)^2 - \cdots - (\omega^{n-1})^2}\right)^2\right) \\
&= \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\omega^2 = g,
\end{aligned}$$

Finally, we finish the proof, and note $\omega^n = \sqrt{1-(\omega^1)^2 - \cdots - (\omega^{n-1})^2}$. ♣

3 Model Riemannian Manifolds

3.1 Notes

Definition 3.1.1

A Riemannian manifold (M, g) is called a **homogeneous Riemannian manifold** if $\text{Iso}(M, g)$ acts transitively on M, i.e., for each pair of points $p, q \in M$, there is an isometry $\varphi : M \to M$ such that $\varphi(p) = q$.

3.2 Grometry of Lie Groups

4 Connections

4.1 Notes

4.2 Some Calculations

$\nabla : T^{(k,l)}TM \to T^{(k,l+1)}TM$, with $\nabla F(\cdots, X) = \nabla_X F$ and

$$\begin{aligned}&(\nabla_X F)(\omega^1, \cdots, \omega^k, X_1, \cdots, X_l)\\ =& X(F(\omega^1, \cdots, \omega^k, X_1, \cdots, X_l))\\ &- \sum_{j=1}^{l} F(\omega^1, \cdots, \nabla_X \omega^i, \cdots, \omega^k, X_1, \cdots, X_l)\\ &- \sum_{i=1}^{k} F(\omega^1, \cdots, \omega^k, X_1, \cdots, \nabla_X X_j, \cdots, X_l).\end{aligned}$$

We use the notation $F^{i_1\cdots i_k}_{j_1\cdots j_l;m}$ denotes the m compoent of ∇F, and

$$\nabla^2_{X,Y} F := \nabla^2 F(\cdots, Y, X),$$

we now show that $\boxed{\nabla^2_{X,Y} F = \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F}$.

Proof. We firstly show that

$$\nabla_Y F = \mathrm{tr}(\nabla F \otimes Y),$$

WLOG we assume $Y = E_m$, then we have $\nabla_{E_m} F = F^{i_1\cdots i_k}_{j_1\cdots j_l;m}$, and since

$$(\nabla F \otimes E_m)^{i_1\cdots i_k m}_{j_1\cdots j_l j_{l+1}} = F^{i_1\cdots i_k}_{j_1\cdots j_l;j_{l+1}},$$

thus we have $\mathrm{tr}(F \otimes Y) = F^{i_1\cdots i_k}_{j_1\cdots j_l;m}$, so we know that they agree. Then the last is trivial. ♣

Now for Hessian of u, $\nabla^2 u$ is a $(0,2)$ tensor, $\nabla^u = u_{;ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$, and

$$\nabla^2 u(Y, X) = \nabla_X(\nabla_Y u) - \nabla_{\nabla - XY} u = X(Y(u)) - (\nabla_X Y)u,$$

so we have $u_{;ij} = \partial_i(\partial_j u) - (\Gamma^k_{ij})\partial_k u = \boxed{\partial_i \partial_j u - \Gamma^k_{ij} \partial_k u}$.

Remiander

1. Propsition 2.9 and Exercise 2.10, I don't know how to describe the topology of UTM, although it is sphere bundle over M;