2023 Fall Partial Differential Equations Rieview 1: Wave Equations

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Abstract

This is a brief rieview of Chapter 1-4 of [1] and the 2023 Fall PDE courses taught by G,H Hu of first three weeks.

Contents

1 Vector Calculus

Theorem 1.1 (Divergence Theorem)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1 boundary, for a vector field $\boldsymbol{F} \in C^1(\bar{\Omega}, \mathbb{R}^n)$, then*

$$\int_\Omega \nabla \cdot \boldsymbol{F} \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{v} \mathrm{d}S, \tag{1.1}$$

where $\boldsymbol{v}$ is the outward unit normal to $\partial\Omega$.

Corollary 1.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1 boundary, and $u \in C^2(\bar{\Omega})$, then*

$$\int_\Omega \Delta u \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{v}} \mathrm{d}S. \tag{1.2}$$

Proof. This is from $\Delta u = \nabla \cdot \nabla u$, and $\nabla u \cdot \boldsymbol{v} = \frac{\partial u}{\partial \boldsymbol{v}}$. □

Corollary 1.3 (Integral by Part)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1 boundary, and $u \in C^2(\bar{\Omega})$, and $v \in C^1(\bar{\Omega})$,*

$$\int_\Omega [\nabla u \cdot \nabla v + u \cdot \Delta v] \mathrm{d}x = \int_{\partial\Omega} u \frac{\partial v}{\partial \boldsymbol{v}} \mathrm{d}S. \tag{1.3}$$

2 Conservation Equations and Characteristics

Here we offer a detailed process to solve 1-conservation equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + v \dfrac{\partial u}{\partial x} + w = 0, \\ u(0, x) = g(x). \end{cases} \tag{2.1}$$

1. Solve ODE and find characteristic $x(t)$ such that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(t, x(t)). \tag{2.2}$$

2. From Lagrangian derivative sovle another ODE:

$$\frac{Du(t,x(t))}{Dt} = w(t,x(t),u(t,x(t))). \tag{2.3}$$

3. Using initial condition to transfer $u(t,x(t))$ to $u(t,x)$, the principal of this approach comes from the observation that $u-\int w$ is constant on the characteristic line, so we can find the point which is the intersection of the time line and the characteristic line.

3 The Wave Equation

Theorem 3.1 (d'Alembert's Formula)**.** *Under the initial conditions*

$$u(0,x) = g(x), \quad \frac{\partial u}{\partial t}(0,x) = h(x), \tag{3.1}$$

for $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, then the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{3.2}$$

admits a unique solution

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau. \tag{3.3}$$

Remark 3.2. *If we add a boundary condition*

$$u(t,0) = u(t,\ell) = 0, \tag{3.4}$$

the wave equation (3.2) *with initial conditions* (3.1)*, then we have the same formula as* (3.3) *for $\widetilde{g}$ and $\widetilde{h}$, where $\widetilde{f}$ denotes the even and 2ℓ periodic extension of f.*

Theorem 3.3 (Inhomogeneous Wave Equations)**.** *For $f \in C^1(\mathbb{R}_+ \times \mathbb{R})$, the unique solution of the wave equation with forcing terms*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = f(t,x), \\ u(0,x) = \dfrac{\partial u}{\partial t}(0,x) = 0 \end{cases} \tag{3.5}$$

is given by

$$u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}s\mathrm{d}y. \tag{3.6}$$

Theorem 3.4 (Kirchhoff's Integral Formula for 3D Wave Equation)**.** *For $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^3)$, suppose that*

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \tag{3.7}$$

under the initial conditions

$$u|_{t=0} = g, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = h, \tag{3.8}$$

then

$$u(t,\boldsymbol{x}) = \frac{\partial}{\partial}\widetilde{g}(\boldsymbol{x};t) + \widetilde{h}(\boldsymbol{x};t), \tag{3.9}$$

with $\widetilde{f}$ defined as

$$\widetilde{f}(\boldsymbol{x};\rho) := \frac{1}{4\pi\rho}\int_{\partial B(\boldsymbol{x},\rho)} f(\boldsymbol{w})\mathrm{d}S(\boldsymbol{w}). \tag{3.10}$$

Theorem 3.5 (Poisson's Integral Formula for 2D)**.** *For $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^2)$, suppose that*

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \tag{3.11}$$

under the initial conditions

$$u|_{t=0} = g, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = h, \tag{3.12}$$

then

$$u(t, \boldsymbol{x}) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{\mathbb{D}} \frac{g(\boldsymbol{x} - t\boldsymbol{y})}{\sqrt{1 - |\boldsymbol{y}|^2}} \mathrm{d}\boldsymbol{y}\right) + \frac{t}{2\pi}\int_{\mathbb{D}} \frac{h(\boldsymbol{x} - t\boldsymbol{y})}{\sqrt{1 - |\boldsymbol{y}|^2}} \mathrm{d}\boldsymbol{y}. \tag{3.13}$$

Theorem 3.6 (Energy Method and Uniqueness)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary. A solution $u \in C^2(\mathbb{R}_+ \times \bar{\Omega})$ of the equation*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f, \quad u|_{\partial\Omega} = 0, \\ u|_{t=0} = g, \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = h, \end{cases} \tag{3.14}$$

is uniquely determined by the functions f, g, h.

Proof. The key point is to define the energy

$$\mathscr{E}[u] := \frac{1}{2}\int_{\Omega} \left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2|\nabla u|^2\right] \mathrm{d}\boldsymbol{x}, \tag{3.15}$$

then show it is contant independent of the time t. □

References

[1] D. Borthwick, *Introduction to partial differential equations.* Springer, 2017.