2023 Fall Partial Differential Equations Exercise 5: Distributions

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1 Problem 12.1

Problem. *Define the distribution $u \in \mathscr{D}'(\mathbb{R})$ by*

$$(u, \psi) := \int_{-1}^{1} \frac{\psi(x) - \psi(0)}{x} \mathrm{d}x + \int_{|x| \geq 1} \frac{\psi(x)}{x} \mathrm{d}x.$$

Show that $u = \mathrm{PV}[x^{-1}]$.

Proof. For $\psi \in C_{\mathrm{cpt}}^{\infty}(\mathbb{R})$, we need to prove

$$(u, \psi) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\psi(x)}{x} \mathrm{d}x.$$

Note that when $\varepsilon < 1$, we have

$$\int_{|x| \geq \varepsilon} \frac{\psi(x)}{x} \mathrm{d}x = \int_{|x| \geq 1} \frac{\psi(x)}{x} \mathrm{d}x + \left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \right) \frac{\psi(x)}{x} \mathrm{d}x.$$

Since ψ is differentiable at $x = 0$, $\frac{\psi(x)-\psi(0)}{x}$ is bounded when $x \in [-1, 1]$. Hence

$$\int_{-1}^{1} \frac{\psi(x) - \psi(0)}{x} \mathrm{d}x$$

exists. We also note that

$$\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \right) \frac{\psi(0)}{x} \mathrm{d}x = \psi(0)(\log \varepsilon - \log \varepsilon) = 0,$$

thus we have

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\psi(x)}{x} \mathrm{d}x &= \lim_{\varepsilon \to 0^+} \left[\int_{|x| \geq 1} \frac{\psi(x)}{x} \mathrm{d}x + \left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \right) \frac{\psi(x) - \psi(0)}{x} \mathrm{d}x \right] \\ &= \int_{-1}^{1} \frac{\psi(x) - \psi(0)}{x} \mathrm{d}x + \int_{|x| \geq 1} \frac{\psi(x)}{x} \mathrm{d}x. \end{aligned}$$

Hence we complete the proof. □

2 Problem 12.2

Problem. *Let $f \in L^1_{\text{loc}}(\mathbb{R})$ be the function*

$$f(x) = \begin{cases} \log x, & x > 0, \\ -\log(-x), & x < 0. \end{cases}$$

Show that the distributional derivative is

$$(f', \psi) = \int_{-1}^{1} \frac{\psi(x) - \psi(0)}{|x|} \mathrm{d}x + \int_{|x| \geq 1} \frac{\psi(x)}{|x|} \mathrm{d}x.$$

Proof. By the definition of the distributional derivative, we have

$$(f', \psi) = -(f, \psi') = -\int_{0}^{+\infty} \psi'(x) \log x \mathrm{d}x + \int_{-\infty}^{0} \psi'(x) \log(-x) \mathrm{d}x.$$

Since ψ has compact support, there exixts $A > 0$, $\text{supp}\psi \subseteq (-A, A)$,

$$-\int_{1}^{+\infty} \psi'(x) \log x \mathrm{d}x = -\int_{1}^{A} \psi'(x) \log x \mathrm{d}x = \psi(x) \log x|_1^A + \int_{1}^{A} \frac{\psi(x)}{x} \mathrm{d}x = \int_{1}^{+\infty} \frac{\psi(x)}{x} \mathrm{d}x.$$

Similarly, we have

$$\int_{-\infty}^{-1} \psi'(x) \log(-x) \mathrm{d}x = \int_{-\infty}^{-1} \frac{\psi(x)}{|x|} \mathrm{d}x.$$

Note that

$$\begin{aligned} & -\int_{0}^{1} \psi'(x) \log x \mathrm{d}x + \int_{-1}^{0} \psi'(x) \log(-x) \mathrm{d}x \\ &= \lim_{\varepsilon \to 0^+} \left(-\int_{\varepsilon}^{1} \psi'(x) \log x \mathrm{d}x + \int_{-1}^{-\varepsilon} \psi'(x) \log(-x) \mathrm{d}x \right) \\ &= \lim_{\varepsilon \to 0^+} \left[\left(\int_{\varepsilon}^{1} + \int_{-1}^{-\varepsilon} \right) \frac{\psi(x)}{|x|} \mathrm{d}x + \psi(\varepsilon) \log \varepsilon + \psi(-\varepsilon) \log \varepsilon \right] \\ &= \int_{-1}^{1} \frac{\psi(x) - \psi(0)}{|x|} \mathrm{d}x \\ &\quad + \lim_{\varepsilon \to 0^+} \left[-\left(\int_{\varepsilon}^{1} + \int_{-1}^{-\varepsilon} \right) \frac{\psi(0)}{|x|} \mathrm{d}x + \psi(\varepsilon) \log \varepsilon + \psi(-\varepsilon) \log \varepsilon \right]. \end{aligned}$$

Where we use ψ is differentiable at 0, $\frac{\psi(x) - \psi(0)}{|x|}$ is bounded. Suppose $\max_{|x| \leq 1} |\psi'(x)| = M$, then we have

$$\begin{aligned} & \left| -\left(\int_{\varepsilon}^{1} + \int_{-1}^{-\varepsilon} \right) \frac{\psi(0)}{|x|} \mathrm{d}x + \psi(\varepsilon) \log \varepsilon + \psi(-\varepsilon) \log \varepsilon \right| \\ &= |(\psi(\varepsilon) - \psi(0)) \log \varepsilon + (\psi(-\varepsilon) - \psi(0)) \log \varepsilon| \\ &\leq 2M\varepsilon \cdot \log \varepsilon \to 0, \quad \text{as } \varepsilon \to 0^+. \end{aligned}$$

Hence we have

$$-\int_{0}^{1} \psi'(x) \log x \mathrm{d}x + \int_{-1}^{0} \psi'(x) \log(-x) \mathrm{d}x = \int_{-1}^{1} \frac{\psi(x) - \psi(0)}{|x|} \mathrm{d}x,$$

and finally we complete the proof. □

3 Problem 12.3

Problem. *Let $\mathbb{H}$ denote the upper half-plane $\{x_2 > 0\} \subseteq \mathbb{R}^2$. The goal of this problem is to show that the Laplace equation on $\mathbb{H}$,*

$$\Delta u = 0, \quad u(\cdot, 0) = g,$$

has the solution

$$u(\boldsymbol{y}) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y_2}{(x - y_1)^2 + y_2^2} g(x) \mathrm{d}x$$

for $g \in C^\infty_{\mathrm{cpt}}(\mathbb{R})$.

1. *Derive this formula from Theorem 12.10 of [1] using the method of images as in Example 12.11. In this case the reflection of $\boldsymbol{y} \in \mathbb{H}$ is given by $\overline{(y_1, y_2)} = (y_1, y_2)$ (the complex conjugate).*
2. *Show that the fact that $u(\cdot, 0) = g$ could also be derived by using lemma 12.1 of [1] to deduce that*

$$\lim_{x \to 0} \frac{y}{\pi(x^2 + y^2)} = \delta(y).$$

Proof. From theorem 12.10 of [1], we have

$$u(\boldsymbol{y}) = -\int_{\partial \mathbb{H}} g(x_1) \frac{\partial G_{\boldsymbol{y}}(x_1)}{\partial \upsilon} \mathrm{d}x_1 = \int_{x_2 = 0} g(x_1) \frac{\partial G_{\boldsymbol{y}}(x_1)}{\partial x_2} \mathrm{d}x_1,$$

where $G_{\boldsymbol{y}}$ is the Green's function of $\mathbb{H}$. We define $\widetilde{G}_y(x) = \Phi_y(x) - \Phi_{\widetilde{y}}(x) = -\frac{1}{2\pi} \log \frac{|\boldsymbol{x}-\boldsymbol{y}|}{|\boldsymbol{x}-\widetilde{\boldsymbol{y}}|}$. Now it is clear that

$$\begin{cases} -\Delta \widetilde{G}_{\boldsymbol{y}} = -\Delta \Phi_{\boldsymbol{y}} - \Delta \Phi_{\widetilde{\boldsymbol{y}}} = \delta(\boldsymbol{x} - \boldsymbol{y}), \\ \widetilde{G}_{\boldsymbol{y}}(\boldsymbol{x}) = 0, \quad \text{on } x_1 = 0. \end{cases}$$

Hence $\widetilde{G}_{\boldsymbol{y}}(\boldsymbol{x}) = G_{\boldsymbol{y}}(\boldsymbol{x})$ is the Green's function. Thus we have

$$u(y_1, y_2) = -\frac{1}{2\pi} \int_{\mathbb{R}} g(x_1) \frac{\partial}{\partial x_2} \left(\log |\boldsymbol{x} - \boldsymbol{y}| - \log |\boldsymbol{x} - \widetilde{\boldsymbol{y}}| \right) \mathrm{d}x_1.$$

Since

$$\frac{\partial}{\partial x_2} \log |\boldsymbol{x} - \boldsymbol{y}| = \frac{x_2 - y_2}{|\boldsymbol{x} - \boldsymbol{y}|^2}, \quad \frac{\partial}{\partial x_2} \log |\boldsymbol{x} - \widetilde{\boldsymbol{y}}| = \frac{x_2 + y_2}{|\boldsymbol{x} - \widetilde{\boldsymbol{y}}|^2},$$

we have

$$\left(\frac{\partial}{\partial x_2} \left(\log |\boldsymbol{x} - \boldsymbol{y}| - \log |\boldsymbol{x} - \widetilde{\boldsymbol{y}}| \right) \right) \Bigg|_{x_2 = 0} = \frac{-2y_2}{|\boldsymbol{x} - \boldsymbol{y}|^2} \Bigg|_{x_2 = 0} = \frac{-2y_2}{(x_1 - y_1)^2 + y_2^2}.$$

Hence we know that

$$u(y_1, y_2) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y_2}{(x - y_1)^2 + y_2^2} g(x) \mathrm{d}x.$$

Now we only need to check $u(\cdot, 0) = g$. By direct calculation, we have

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{y_2}{(x - y_1)^2 + y_2^2} \mathrm{d}x = 1,$$

thus

$$|u(y_1, y_2) - g(y_1)| = \frac{1}{\pi} \left| \int_{\mathbb{R}} \frac{y_2}{(x - y_1)^2 + y_2^2} (g(x) - g(y_1)) \mathrm{d}x \right| \to 0$$

as $y_2 \to 0$. Thus we have $u(y_1, 0) = g(y_1)$. Finally we finish the proof.

(2)Recall in lemma 12.1 of [1], if $\int_{\mathbb{R}} f(x_1) \mathrm{d}x_1 < +\infty$, then $af(ax_1) \to \delta(x_1)$ as $a \to +\infty$. Take

$$f(x_1) = \frac{1}{\pi(1 + x_1^2)},$$

and choose $a = \frac{1}{y_2}$ as $y_2 \to 0$. Then we have

$$\frac{1}{y_2} \cdot \frac{1}{\pi\left(1 + \left(\frac{x_1}{y_2}\right)^2\right)} = \frac{1}{\pi} \cdot \frac{y_2}{y_2^2 + x_1^2} \to \delta(x_1).$$

Thus

$$u(y_1, y_2) \overset{y_2 \to 0}{\longrightarrow} \int_{\mathbb{R}} \delta(x_1 - y_1) g(x_1) \mathrm{d}x_1 = (\delta * g)(y_1) = g(y_1).$$

Finally, we finish the proof. □

4 Problem 12.4

Problem. *In* $\mathbb{R}^3$ *show that*

$$(-\Delta - k^2)\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} = \delta$$

for all $k \in \mathbb{R}$.

Proof. For $\psi \in C^\infty_{\mathrm{cpt}}(\mathbb{R})$, we need to prove

$$\int_{\mathbb{R}^3} (-\Delta - k^2)\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} \cdot \psi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \psi(\boldsymbol{0}).$$

Since $\left(\Delta \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}, \psi\right) = \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}, \Delta\psi\right)$, we have

$$\int_{\mathbb{R}^3} \Delta \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} \cdot \psi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} \cdot \Delta\psi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Note that ψ is compactly supported, hence we have

$$\begin{aligned}
-\int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} \cdot \Delta\psi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^3} \nabla \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} \cdot \nabla\psi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \frac{(\mathrm{i}kr - 1)\mathrm{e}^{\mathrm{i}kr}}{4\pi r^3} \boldsymbol{x} \cdot \nabla\psi \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \frac{(\mathrm{i}kr - 1)\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2} \frac{\partial\psi}{\partial r} \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{S}^2} \mathrm{d}S \int_0^\infty \frac{(\mathrm{i}kr - 1)\mathrm{e}^{\mathrm{i}kr}}{4\pi} \frac{\partial\psi}{\partial r} \mathrm{d}r \\
&= \int_{\mathbb{S}^2} \left(\frac{\psi(\boldsymbol{0})}{4\pi} - \int_0^\infty \frac{-k^2 r \mathrm{e}^{\mathrm{i}kr}}{4\pi} \psi \mathrm{d}r\right) \mathrm{d}S \\
&= \psi(\boldsymbol{0}) - \int_{\mathbb{S}^2} \mathrm{d}S \int_0^\infty \frac{-k^2 r \mathrm{e}^{\mathrm{i}kr}}{4\pi} \psi \mathrm{d}r \\
&= \psi(\boldsymbol{0}) - \int_{\mathbb{R}^3} \frac{-k^2 \mathrm{e}^{\mathrm{i}kr}}{4\pi r} \psi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Hence we have

$$\int_{\mathbb{R}^3} -\Delta\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\right)\psi - k^2 \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} \psi \mathrm{d}\boldsymbol{x} = \psi(\boldsymbol{0}),$$

then we finish the proof. □

References

[1] D. Borthwick, *Introduction to partial differential equations.* Springer, 2017.