2023 Fall Partial Differential Equations Exercise 3: Fourier Series

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Contents

1 Problem 8.1

Problem. *For $x \in (0, \pi)$, let*

$$f(x) = x,$$

(a) Extend f to an odd function on $\mathbb{T}$ and compute the periodic Fourier coefficients;

(b) Show that the convergence of the Fourier series at $x = \frac{\pi}{2}$ yields the summation formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

(c) Show the Parseval identity leads to the formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Solution. (a) Extending f to an odd function on $\mathbb{T}$, we have $f(x) = x$ when $x \in (-\pi, \pi)$. Hence,

$$\begin{aligned}
c_0[f] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \mathrm{d}x = 0, \\
c_k[f] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\cos kx - \mathrm{i} \sin kx) \mathrm{d}x \\
&= \frac{\mathrm{i}}{2\pi} \left(\frac{-x \cos kx}{k} \bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos kx}{k} \mathrm{d}x \right) = \frac{(-1)^{k+1} \cdot \mathrm{i}}{k}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
S_n[f](x) &= \sum_{k=-n}^{n} c_k[f] \mathrm{e}^{\mathrm{i}kx} = \sum_{k=-n}^{n} \frac{(-1)^{k+1} \cdot \mathrm{i}}{k} (\cos kx + \mathrm{i} \sin kx) \\
&= \sum_{k=1}^{n} \left(\frac{(-1)^{k+1} \cdot \mathrm{i}}{k} + \frac{(-1)^{k+1} \cdot \mathrm{i}}{-k} \right) \cos kx + \sum_{k=1}^{n} \left(\frac{(-1)^{k+1}}{k} + \frac{(-1)^{k+1}}{k} \right) \sin kx \\
&= \sum_{k=1}^{n} \frac{(-1)^{k+1} \cdot 2}{k} \sin kx.
\end{aligned}$$

(b) From theorem 8.3 of [1] and $f(x) = x$ is differentiable at $x = \frac{\pi}{2}$, we have

$$\lim_{n\to\infty} S_n[f]\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right).$$

Hence,

$$\frac{\pi}{4} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin\frac{k\pi}{2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

(c)From corollary 8.7 of [1] and $f \in L^2(\mathbb{T})$, we have

$$\sum_{k\in\mathbb{Z}} |c_k[f]|^2 = \frac{1}{2\pi}||f||_{L^2}.$$

Since

$$\sum_{k\in\mathbb{Z}} |c_k[f]|^2 = \sum_{k\in\mathbb{Z}^*} \frac{1}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2},$$

and

$$\frac{1}{2\pi}||f||_{L^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x = \frac{1}{2\pi} \cdot \frac{2\pi^3}{3} = \frac{\pi^2}{3}.$$

Then we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Finally we finish this problem. □

2 Problem 8.2

Problem. *For $x \in (0, \pi)$, let*

$$f(x) = x,$$

(a) Extend f to a even function on $\mathbb{T}$ and compute the periodic Fourier coefficients;

(b) Show that the convergence of the Fourier series at $x = 0$ yields the summation formula

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

(c) Show the Parseval identity leads to the formula

$$\sum_{k\in\mathbb{N}_{\text{odd}}} \frac{1}{k^4} = \frac{\pi^4}{96}.$$

Solution. (a) Extending g to an even function on $\mathbb{T}$, we have $g(x) = |x|$ when $x \in (-\pi, \pi)$. Hence,

$$\begin{aligned} c_0[f] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| \mathrm{d}x = \frac{\pi}{2}, \\ c_k[f] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|(\cos kx - \mathrm{i}\sin kx)\mathrm{d}x \\ &= \frac{1}{\pi}\left(\frac{x\sin kx}{k}\bigg|_0^{\pi} - \int_0^{\pi} \frac{\sin kx}{k}\mathrm{d}x\right) = \frac{(-1)^k - 1}{k^2\pi}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} S_n[f](x) &= \sum_{k=-n}^{n} c_k[f]\mathrm{e}^{\mathrm{i}kx} = \sum_{k=-n}^{n} \frac{(-1)^k - 1}{k^2\pi}(\cos kx + \mathrm{i}\sin kx) \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(k-1)x}{(2k-1)^2}. \end{aligned}$$

(b) At $x = 0$ and for $\varepsilon = 1$, we have

$$\text{ess} - \sup_{y\in[-1,1]} \left| \frac{f(x) - f(x-y)}{y} \right| = \text{ess} - \sup_{y\in[-1,1]} \left| \frac{|x| - |x-y|}{y} \right| \leq 1.$$

Now From theorem 8.3 of [1], we have

$$\lim_{n\to\infty} S_n[g]\,(0) = g\,(0)\,.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

(c)From corollary 8.7 of [1] and $g \in L^2(\mathbb{T})$, we have

$$\sum_{k\in\mathbb{Z}} |c_k[g]|^2 = \frac{1}{2\pi} ||g||_{L^2}.$$

Since

$$\sum_{k\in\mathbb{Z}} |c_k[g]|^2 = \frac{\pi^2}{4} + \sum_{k\in\mathbb{N}_{\text{odd}}} \frac{8}{k^4\pi^2},$$

and

$$\frac{1}{2\pi}||g||_{L^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x = \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} = \frac{\pi^2}{3}.$$

Then we have

$$\sum_{k\in\mathbb{N}_{\text{odd}}} \frac{1}{k^4} = \frac{\pi^4}{96}.$$

Finally we finish this problem. □

3 Problem 8.3

Problem. *Consider the periodic wave equation*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

for $t \in \mathbb{R}$ and $x \in \mathbb{T}$. Suppose the initial conditions are

$$u(0,x) = g(x), \quad \frac{\partial u}{\partial t}(0,x) = h(x),$$

for $g \in C^{m+1}(\mathbb{T})$ and $h \in C^m(\mathbb{T})$, for $m \in \mathbb{N}$.

(a) Assuming that $u(t,x)$ can be represented as a Fourier series

$$u(t,x) = \sum_{k\in\mathbb{Z}} a_k(t)\mathrm{e}^{\mathrm{i}kx}, \tag{3.1}$$

find an expression for $a_k(t)$ in terms of the Fourier coefficients of g and h.

(b) Using the assumptions on g and h, show that the coefficients $a_k(t)$ satisfy an estimate

$$\sum_{k\in\mathbb{Z}} k^{2m}|a_k(t)|^2 \leq M < \infty,$$

uniformly for $t \in \mathbb{R}$.

(c) What could you conclude about the differentiability of u?

Proof. (a) We extend g and h to be functions defined on $\mathbb{R}$. From theorem 4.1 of [1], we have

$$u(t,x)=\frac{1}{2}[g(x+t)+g(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}h(\tau)\mathrm{d}\tau. \tag{3.2}$$

Suppose

$$g(x)=\sum_{k\in\mathbb{Z}}c_k[g]\mathrm{e}^{\mathrm{i}kx},\quad h(x)=\sum_{k\in\mathbb{Z}}c_k[h]\mathrm{e}^{\mathrm{i}kx}.$$

Since $h\in C^m(\mathbb{T})$, we can interchange the integral and summation from Lebesgue dominated convergence theorem. Hence we have

$$\begin{aligned}u(t,x)&=\sum_{k\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}kx}\left(\frac{1}{2}c_k[g](\mathrm{e}^{\mathrm{i}kt}+\mathrm{e}^{-\mathrm{i}kt})\right)+\sum_{k\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}kx}\left(\frac{1}{2\mathrm{i}k}c_k[h](\mathrm{e}^{\mathrm{i}kt}-\mathrm{e}^{-\mathrm{i}kt})\right)\\&=\sum_{k\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}kx}\left(c_k[g]\cos kt+c_k[h]\cdot\frac{\sin kt}{k}\right).\end{aligned}$$

Hence we have

$$\boxed{a_k(t)=c_k[g]\cos kt+c_k[h]\cdot\frac{\sin kt}{k}}.$$

(b) From Cauchy-Schwarz inequality, we have

$$|a_k(t)|^2=\left(c_k[g]\cos kt+c_k[h]\cdot\frac{\sin kt}{k}\right)^2\le\left(|c_k[g]|^2+\frac{|c_k[h]|^2}{k^2}\right)(\cos^2 kt+\sin^2 kt)=|c_k[g]|^2+\frac{|c_k[h]|^2}{k^2}.$$

Now from theorem 8.10 of [1], and $g\in C^{m+1}(\mathbb{T})$ and $h\in C^m(\mathbb{T})$. We have

$$\sum_{k\in\mathbb{Z}}k^{2m}|c_k[g]|^2=M_1<\infty,\quad\sum_{k\in\mathbb{Z}}k^{2m-2}|c_k[h]|^2=M_2<\infty.$$

Thus

$$\sum_{k\in\mathbb{Z}}k^{2m}|a_k(t)|^2\le\sum_{k\in\mathbb{Z}}\left(k^{2m}|c_k[g]|^2+k^{2m-2}|c_k[h]|^2\right)=M_1+M_2<\infty,$$

uniformly for $t\in\mathbb{R}$.

(3) From Cauchy-Schwarz inequality, we have

$$\left(\sum_{k\in\mathbb{Z}^*}|k^{m-1}a_k(t)|\right)^2\le\left(\sum_{k\in\mathbb{Z}}k^{2m}|a_k(t)|^2\right)\cdot\left(\sum_{k\in\mathbb{Z}^*}\frac{1}{k^2}\right)\le\frac{\pi^2(M_1+M_2)}{3}<\infty.$$

Hence, from $\sum_{k\in\mathbb{Z}}|k^{m-1}a_k(t)|<\infty$ and theorem 8.12 of [1], we have $u(t,\cdot)\in C^{m-1}(\mathbb{T})$ for $t>0$. □

References

[1] D. Borthwick, *Introduction to partial differential equations.* Springer, 2017.