2023 Fall Partial Differential Equations Exercise 2: Heat Equations

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Abstract

I select five problems to solve from [1].

Contents

1 Problem 6.1

Problem. *Find a formula for the the reaction-diffusion equations*

$$\frac{\partial u}{\partial t} + \gamma u - \Delta u = 0 \tag{1.1}$$

on $\mathbb{R}^n$ *with initial condition* $u(0, \boldsymbol{x}) = f(\boldsymbol{x})$*, where we assume* f *continuous and bounded.*

Solution. Let $u(t, \boldsymbol{x}) = \mathrm{e}^{-\gamma t} w(t, \boldsymbol{x})$, then by direct calculation, we have

$$0 = -\gamma \mathrm{e}^{-\gamma t} w + \mathrm{e}^{-\gamma t} \frac{\partial w}{\partial t} + \gamma \mathrm{e}^{-\gamma t} w - \mathrm{e}^{-\gamma t} \Delta w,$$

so since $\mathrm{e}^{-\gamma t} \neq 0$, we can transfer (1.1) to the heat equation below

$$\frac{\partial w}{\partial t} - \Delta w = 0 \tag{1.2}$$

with initial condition $w(0, \boldsymbol{x}) = f(\boldsymbol{x})$, so we have a classical solution for (1.2):

$$w(t, \boldsymbol{x}) = H_t * f(\boldsymbol{x}) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{y}|^2/4t} f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{1.3}$$

then we have

$$\boxed{u(t, \boldsymbol{x}) = (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-\gamma t} \int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{y}|^2/4t} f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}}$$

is a soulution of (1.1). □

2 Problem 6.3

Problem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise C^1 boundary. Suppose that $u(t,x)$ satisfies the heat equation*

$$\frac{\partial u}{\partial t} - \Delta u = 0, \tag{2.1}$$

on $(0,\infty)\times\Omega$, we define the total thermal energy at time t by

$$\mathscr{U}[t] = \int_\Omega u(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{2.2}$$

(a) Assume that u satisfies Neumann boundary conditions,

$$\left.\frac{\partial u}{\partial \boldsymbol{v}}\right|_{\partial\Omega} = 0, \tag{2.3}$$

show that $\mathscr{U}$ is constant.

(b) Assume that u is positive in the interior of Ω and equals 0 on the boundary. Show that $\mathscr{U}(t)$ is decreasing in this case.

Proof. (a) Since Ω is bounded, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}[t] = \int_\Omega \frac{\partial u}{\partial t}\mathrm{d}\boldsymbol{x} = \int_\Omega \Delta u \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega}\frac{\partial u}{\partial \boldsymbol{v}}\mathrm{d}S = 0, \tag{2.4}$$

then we know that $\mathscr{U}[t] \equiv \mathscr{U}[0]$ is a constant.

(b) As (2.4) shows, we still have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}[t] = \int_{\partial\Omega}\frac{\partial u}{\partial \boldsymbol{v}}\mathrm{d}S, \tag{2.5}$$

then since $\boldsymbol{v}$ is the outward unit normal vector, so by definition, for any $\boldsymbol{x}\in\partial\Omega$, $\boldsymbol{x}+t\boldsymbol{v}\in\Omega$ when $t<0$, then from $u(\boldsymbol{x}+t\boldsymbol{v})>0$ and $u(\boldsymbol{x})=0$, so we have $u(\boldsymbol{x}+t\boldsymbol{v})-u(\boldsymbol{x})>0$, while $t\to 0^-$, then

$$\frac{\partial u}{\partial \boldsymbol{v}}(\boldsymbol{x}) = \lim_{t\to 0^-}\frac{u(\boldsymbol{x}+t\boldsymbol{v})-u(\boldsymbol{x})}{t} \le 0, \tag{2.6}$$

so the integral of (2.5) is nonpositive, so we have $\mathscr{U}[t]$ is decreasing. □

3 Problem 6.4

Problem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise C^1 boundary. Suppose that $u(t,x)$ satisfies the heat equation*

$$\frac{\partial u}{\partial t} - \Delta u = 0, \tag{3.1}$$

on $(0,\infty)\times\Omega$. Define

$$\eta(t) := \int_\Omega u(t,\boldsymbol{x})^2\mathrm{d}\boldsymbol{x}. \tag{3.2}$$

(a) Assume that u satisfies the Dirichlet boundary conditions:

$$u(t,\boldsymbol{x})|_{\boldsymbol{x}\in\partial\Omega} = 0 \tag{3.3}$$

for $t\ge 0$. Show that η decreases as a function of t.

(b) Use (a) to show that a solution u satisfying boundary and initial conditions

$$u|_{t=0} = g, \quad u|_{\boldsymbol{x}\in\partial\Omega} = h, \tag{3.4}$$

for some continuous functions g on Ω and h on $\partial\Omega$, is uniquely determined by g and h.

Proof. (a) Since Ω is bounded, we have

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = 2\int_\Omega u \cdot \frac{\partial u}{\partial t}\mathrm{d}\boldsymbol{x} = 2\int_\Omega u \cdot \Delta u \mathrm{d}\boldsymbol{x} = -2\int_\Omega |\nabla u|^2 \mathrm{d}\boldsymbol{x} + 2\int_{\partial\Omega} u\frac{\partial u}{\partial \boldsymbol{v}}\mathrm{d}S = -2\int_\Omega |\nabla u|^2 \mathrm{d}\boldsymbol{x} \le 0, \tag{3.5}$$

so we have η decreases as a function of t.

(b) Suppose u_1 and u_2 are solutions of (3.1) with (3.4), then we have $u_1 - u_2$ is a solution of (3.1) with

$$u|_{t=0} = 0, \quad u|_{\boldsymbol{x}\in\partial\Omega} = 0, \tag{3.6}$$

then from (a) we have for such u, $\eta(t)$ decreases, and since $\eta(0) = 0$, and $\eta(t) \ge 0$ for all t, thus $\eta(t) \equiv 0$, then we have $u \equiv 0$, i.e., $u_1 \equiv u_2$, so we have the solution is uniquely determined by g and h. ☐

4 Problem 7.6

Problem. *Suppose that u solves the heat equation with*

$$u|_{t=T} = 0, \quad u|_{\boldsymbol{x}\in\partial\Omega} = 0. \tag{4.1}$$

The goal is to show that these assumptions imply $u = 0$ for all t.

(a) Use the Cauchy-Schwarz inequality to deduce that

$$\eta'(t)^2 \le 4\eta(t)\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x}, \tag{4.2}$$

where η is defined as in (3.2)

(b) Show that

$$\eta''(t) = 4\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x}, \tag{4.3}$$

so that the inequality from (a) becomes

$$\eta'(t)^2 \le \eta(t)\eta''(t). \tag{4.4}$$

(c) Show that if $\eta(0) > 0$, then η is positive for all $t \ge 0$.

(d) Conclude from (c) that if $\eta(T) = 0$, then $\eta(t) = 0$ for all t, and deduce that $u = 0$.

Proof. (a) From (3.5), we have

$$\eta'(t)^2 = 4\left(\int_\Omega u \cdot \frac{\partial u}{\partial t}\mathrm{d}\boldsymbol{x}\right)^2 \le 4\left(\int_\Omega u^2 \mathrm{d}\boldsymbol{x}\right)\left(\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x}\right) = 4\eta(t)\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x}. \tag{4.5}$$

(b) By direct calculations, we have

$$\eta''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(2\int_\Omega u \cdot \frac{\partial u}{\partial t}\mathrm{d}\boldsymbol{x}\right) = 2\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x} + 2\int_\Omega u\frac{\partial^2 u}{\partial t^2}\mathrm{d}\boldsymbol{x}, \tag{4.6}$$

from heat equation, we have $u_t := \frac{\partial u}{\partial t} = \Delta u$, thus

$$\int_\Omega u\frac{\partial^2 u}{\partial t^2}\mathrm{d}\boldsymbol{x} = \int_\Omega u \cdot \Delta u_t \mathrm{d}\boldsymbol{x} = \int_\Omega u_t \cdot \Delta u \mathrm{d}\boldsymbol{x} + \left(\int_{\partial\Omega} u\frac{\partial u_t}{\partial \boldsymbol{v}} - u_t\frac{\partial u}{\partial \boldsymbol{v}}\mathrm{d}S\right),$$

where we use Green's formula, then from boundary condition (4.1), we have

$$\int_\Omega u\frac{\partial^2 u}{\partial t^2}\mathrm{d}\boldsymbol{x} = \int_\Omega (u_t)^2 \mathrm{d}\boldsymbol{x}, \tag{4.7}$$

finally, we insert (4.7) into the (4.6), then we finish the proof.

(c) Since if $\eta(0) > 0$, then by continuity $\log \eta(t)$ is defined at least in some neighborhood of $t = 0$, from (4.4),

$$(\log \eta(t))'' = \left(\frac{\eta'(t)}{\eta(t)}\right)' = \frac{\eta''(t)\eta(t) - \eta(t)^2}{\eta(t)^2} \geq 0, \tag{4.8}$$

which implies that $\log \eta(t)$ is bounded below by its tangent lines, in particular, we have

$$\log \eta(t) \geq \log \eta(0) + \frac{\eta'(0)}{\eta(0)} t, \tag{4.9}$$

which implies

$$\eta(t) \geq \eta(0) \mathrm{e}^{-ct}, \tag{4.10}$$

for $c = -\eta'(0)/\eta(0)$, thus if $\eta(0) > 0$, then η is positive for all $t \geq 0$.

(d) If $\eta(T) = 0$, then since $\eta(t)$ decreases from Problem 6.4, then we have $\eta(t) \leq 0$ for all $t > T$, and since naturally $\eta(t) \geq 0$, thus for all $t > T$, $\eta(t) = 0$. Now for $t \leq T$, if $\eta(0) > 0$, then from (c), $0 = \eta(T) \geq \eta(0)\mathrm{e}^{-ct} > 0$, which is absurd, so we know that $\eta(0) = 0$, so for all $0 \leq t \leq T$, $\eta(t) = 0$, finally, we deduce that $\eta(t) \equiv 0$ then $u \equiv 0$, then we finish this problem . □

5 Problem 7.8

Problem. *Solve two questions below:*

(a) Show that

$$\phi_n(x) := \sqrt{\frac{2}{\pi}} \sin(nx), \quad n \in \mathbb{N}, \tag{5.1}$$

defines an orthonormal sequence in $L^2(0, \pi)$.

(b) For the function $u \equiv 1$, compute the corresponding expansion coefficients,

$$c_k[1] := \langle 1, \phi_k \rangle, \tag{5.2}$$

then show that $S_n[1] \to 1$ in $L^2(0, \pi)$.

Proof. (a) By direct calculation

$$\langle \phi_n, \phi_n \rangle = \int_0^\pi \frac{2}{\pi} \sin^2(nx) \mathrm{d}x = \frac{2}{\pi} \left(\frac{x}{2} - \frac{\sin(2nx)}{4n} \right) \Bigg|_0^\pi = 1, \tag{5.3}$$

and also we have when $n \neq m$, then

$$\langle \phi_n, \phi_m \rangle = \int_0^\pi \frac{2}{\pi} \sin(nx) \sin(mx) \mathrm{d}x = \frac{2}{\pi} \left(\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \right) \Bigg|_0^\pi = 0, \tag{5.4}$$

thus $\{\phi_n(x)\}$ defines an orthonormal sequence in $L^2(0, \pi)$.

(b) By direct calculation,

$$c_k[1] = \int_0^\pi \sqrt{\frac{2}{\pi}} \sin(kx) \mathrm{d}x = \sqrt{\frac{2}{\pi}} \cdot \frac{1 - (-1)^k}{k}, \tag{5.5}$$

then from the theorem 7.9 of [1], we know that $S_n[1] \to 1$ in $L^2(0, \pi)$ if and only if

$$\sum_{n=1}^\infty |c_k[1]|^2 = \|1\|^2, \tag{5.6}$$

which is equivalent to

$$\frac{2}{\pi} \cdot \sum_{k=1}^\infty \frac{4}{(2k+1)^2} = \pi, \tag{5.7}$$

then form mathematical analysis, we have already known that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$, then we have

$$\sum_{k=1}^\infty \frac{1}{(2k+1)^2} = \sum_{k=1}^\infty \frac{1}{k^2} - \sum_{k=1}^\infty \frac{1}{(2k)^2} = \left(1 - \frac{1}{4}\right) \frac{\pi^2}{6} = \frac{\pi^2}{8}, \tag{5.8}$$

which is exactly (5.7), then we finish the proof. □

References

[1] D. Borthwick, *Introduction to partial differential equations.* Springer, 2017.