2023 Fall Partial Differential Equations Exercise 1: Wave Equations

2021 Chern Class 2113696 Kai Zhu

September 15, 2023

Abstract

I select five problems to solve from [1].

Contents

1 Problem 3.3

Problem. *Assume that u satisfies the linear conservation equation*

$$\frac{\partial u}{\partial t} + 2t\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

for $t \in \mathbb{R}$ and $x \in [0,1]$, suppose the boundary conditions are given by

$$u(t,0) = h_0(t), \quad u(t,1) = h_1(t).$$

Find a relation between h_0 and h_1.

Solution. Firstly, from the ODE $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 2t$, we can solve that $x(t) = t^2 + x_0$ for some constant x_0, then we have

$$\frac{Du(t,x(t))}{Dt} \equiv 0, \quad \Longrightarrow u(t,x(t)) = C$$

for some another constant C depends on x_0, now from the initial conditions, we have $u(t, t^2 + x_0) = C(x_0)$, then when $t^2 + x_0 = 0$, we have $h_0(t) = u(t,0) = C(x_0) = C(-t^2)$, similarly, we have $h_1(t) = u(t,1) = C(x_0) = C(1-t^2)$, so if (1.1) has a solution u, then for each $t_0, t_1 \in \mathbb{R}$, if $-t_0^2 = 1 - t_1^2$, then we have $h_0(t_0) = h_1(t_1)$, in summary

$$\boxed{h_0(t_0) = h_1(t_1), \quad \text{if } t_1^2 - t_0^2 = 1,}$$

is a relation that h_0 and h_1 should satisfy. □

2 Problem 3.7

Problem. *In the mid-19th century, William Hamilton and Carl Jacobi developed a formulation of classical mechanics based on ideas from geometric optics. In this approach the dynamics of a free particle in* $\mathbb{R}$ *are described by a generating function* $u(t,x)$ *satisfying the Hamilton-Jacobi equation:*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0. \tag{2.1}$$

Assume that $u \in C^1([0,\infty) \times \mathbb{R}^n)$ *is a solution of* (2.1). *We define a characteristic of* (2.1) *is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \frac{\partial u}{\partial x}(t, x(t)), \quad x(0) = x_0. \tag{2.2}$$

(a) Assuming that $x(t)$ *solves* (2.1), *use the chain rule to compute* $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$.

(b) Differentiate (2.1) *with respect to* x *and then restrict the result to* $(t, x(t))$, *where* $x(t)$ *solves* (2.2). *Conclude from (a) that to*

$$\frac{\mathrm{d}^2 x}{\mathrm{d}^2 t} = 0.$$

Hence, for some constant v_0 *(which depends on the characteristic),*

$$x(t) = x_0 + v_0 t.$$

(c) Show that the Lagrangian derivative of u *along* $x(t)$ *satisfies*

$$\frac{Du}{Dt} = \frac{1}{2}v_0^2,$$

imply that

$$u(t, x_0 + v_0 t) = u(0, x_0) + \frac{1}{2}v_0^2 t.$$

(d) Use this approach to find the solution $u(t,x)$ *under the initial condition*

$$u(0,x) = x^2.$$

Solution. (a) By direct calculation, we have

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u}{\partial x}(t, x(t))\right) = \frac{\partial^2 u}{\partial x \partial t}(t, x(t)) + \dot{x}(t)\frac{\partial^2 u}{\partial x^2}(t, x(t)),$$

in short we actually have

$$\boxed{\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x^2}.}$$

(b) Differentiate (2.1), we have

$$\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x^2} = 0,$$

trivially, from (a) we have

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0.$$

Then we solve this ODE with initial condition $x(0) = x_0$, there exists a constant v_0 such that $x(t) = x_0 + v_0 t$.

(c) By direct calculation, we have

$$\frac{Du(t, x(t))}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \dot{x}(t) = \frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{1}{2}(\dot{x}(t))^2 = \frac{1}{2}v_0^2,$$

here we use the identity $\frac{\partial u}{\partial x}(t, x(t)) = \dot{x}(t) = v_0$, so from this we know that $u(t, x(t)) = C + \frac{1}{2}v_0^2 t$ for some constant C, then let $t = 0$, we have

$$\boxed{u(t, x_0 + v_0 t) = u(0, x_0) + \frac{1}{2}v_0^2 t.}$$

(d) Since $u(0,x)=x^2$, then we have $u(0,x_0)=x_0^2$, then since $x=x_0+v_0t$, i.e., $v_0=\frac{x-x_0}{t}$, on the other hand, note that $\frac{\partial u}{\partial x}(0,x)=2x$, so we have $\frac{\partial u}{\partial x}(0,x_0)=2x_0$, note that from (2.2), we have

$$v_0=\dot{x}(0)=\frac{\partial u}{\partial x}(0,x(0))=\frac{\partial u}{\partial x}(0,x_0)=2x_0,$$

so we have $v_0=2x_0$ and $x_0=\frac{x}{1+2t}$, so from the equation we get from (c), i.e., $u(t,x_0+v_0t)=x_0^2+\frac{1}{2}v_0^2t$, then

$$\boxed{u(t,x)=\frac{x^2}{1+2t},}$$

is the solution we want to find for the initial condition $u(0,x)=x^2$. □

3 Problem 4.4

Problem. *Consider a string of length ℓ with progagation speed $c=1$, i.e., we consider a wave equation as*

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=f(t,x):=\cos(\omega t)\sin(\omega_k x), \tag{3.1}$$

with $\omega>0$ and $\omega_k:=\frac{k\pi}{\ell}$ and the intial condition

$$u(0,x)=0, \quad \frac{\partial u}{\partial t}(0,x)=0, \tag{3.2}$$

then find the solution $u(t,x)$ including both cases $\omega\neq\omega_k$ and $\omega=\omega_k$.

Solution. By Duhamel's method, we can direct get the expression of $u(t,x)$ by

$$\begin{aligned} u(t,x)=&\frac{1}{2}\int_0^t\int_{x-t+s}^{x+t-s}\cos(\omega s)\sin(\omega_k y)\mathrm{d}y\mathrm{d}s \\ =&\frac{1}{2\omega_k}\int_0^t[\cos(\omega_k(x-t+s))-\cos(\omega_0(x+t-s))]\cos(\omega s)\mathrm{d}s \\ =&\frac{1}{\omega_k}\int_0^t\sin(\omega_k x)\sin(\omega_k(t-s))\cos(\omega s)\mathrm{d}s. \end{aligned} \tag{3.3}$$

Now from the equation above, we have for $\omega\neq\omega_k$ we obtain

$$\boxed{u(t,x)=\frac{\sin(\omega_k x)}{\omega_k^2-\omega^2}[\cos(\omega t)-\cos(\omega_k t)],}$$

and if $\omega=\omega_k$ then we obtain

$$\boxed{u(t,x)=\frac{t}{2\omega_k}\sin(\omega_k x)\sin(\omega_k t),}$$

now we finish this problem. □

4 Problem 4.5

Problem. *The telegraph equation is a variant of the wave equation that describes the propagation of electrical signals in a one-dimensional cable:*

$$\frac{\partial^2 u}{\partial t^2}+a\frac{\partial u}{\partial t}+bu-c^2\frac{\partial^2 u}{\partial x^2}=0, \tag{4.1}$$

where $u(t,x)$ is the line voltage, c is the propagation speed, and $a,b>0$ are determined by electrical properties of the cable (resistance, inductance, etc.). Show that the substitution

$$u(t,x)=\mathrm{e}^{-at/2}w(t,x) \tag{4.2}$$

reduces the telegraph equation to an ordinary wave equation for w, provided a and b satisfy a certain condition. Find the general solution in this case.

Solution. By direct calculation, we have

$$\frac{\partial}{\partial t}\left(\mathrm{e}^{-at/2}w(t,x)\right)=\mathrm{e}^{-at/2}\frac{\partial w}{\partial t}(t,x)-\frac{a}{2}\mathrm{e}^{-at/2}w(t,x),$$

and

$$\frac{\partial^2}{\partial t^2}\left(\mathrm{e}^{-at/2}w(t,x)\right)=\mathrm{e}^{-at/2}\frac{\partial^2 w}{\partial t^2}(t,x)-a\mathrm{e}^{-at/2}\frac{\partial w}{\partial t}(t,x)+\frac{a^2}{4}\mathrm{e}^{-at/2}w(t,x),$$

so now we from (4.1) and have

$$\mathrm{e}^{-at/2}\frac{\partial^2 w}{\partial t^2}+\mathrm{e}^{-at/2}\left(b-\frac{a^2}{4}\right)w-c^2\mathrm{e}^{-at/2}\frac{\partial^2 w}{\partial x^2}=0, \tag{4.3}$$

so when $4b=a^2$, then we have the ordinary wave equation from (4.3),

$$\frac{\partial^2 w}{\partial t^2}-c^2\frac{\partial^2 w}{\partial x^2}=0,. \tag{4.4}$$

So in general, we can transfer a telegrah equation (4.1) with $4b=a^2$ and initial conditions

$$u(0,x)=g(x),\quad \frac{\partial u}{\partial t}(0,x)=h(x), \tag{4.5}$$

to the ordinary wave equation (4.4) with initial conditions

$$w(0,x)=\widetilde{g}(x)=g(x),\quad \frac{\partial w}{\partial t}(0,x)=\widetilde{h}(x)=h(x)+\frac{a}{2}g(x), \tag{4.6}$$

so now from d'Alembert's formula, we have

$$w(t,x)=\frac{1}{2c}[\widetilde{g}(x+ct)+\widetilde{g}(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\widetilde{h}(\tau)\mathrm{d}\tau, \tag{4.7}$$

then we have the general solution of (4.1) with initial conditions (4.5) is

$$\boxed{u(t,x)=\frac{\mathrm{e}^{-at/2}}{2c}[g(x+ct)+g(x-ct)]+\frac{\mathrm{e}^{-at/2}}{2c}\int_{x-ct}^{x+ct}\left(h(\tau)+\frac{a}{2}g(\tau)\right)\mathrm{d}\tau,}$$

then we finish this problem. □

5 Problem 4.9

Problem. *The Klein-Gordon equation in Rn is a variant of the wave equation that appears in relativistic quantum mechanics,*

$$\frac{\partial^2 u}{\partial t^2}-\Delta u+m^2u=0, \tag{5.1}$$

where m is the mass of a particle.

(a) Find a formula for $\omega=\omega(\boldsymbol{k},m)$ under which this equation have solutions of the form

$$u(t,\boldsymbol{x})=\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{5.2}$$

where $\omega\in\mathbb{R}$ and $\boldsymbol{k}\in\mathbb{R}^n$ are constants.

(b) Show that we can define a conserved energy $\mathscr{E}$ for this equation by adding a term proportional to u2 to the integrand in

$$\mathscr{E}(u)=\frac{1}{2}\int_{\Omega}\left[\left(\frac{\partial u}{\partial t}\right)^2+|\nabla u|^2\right]\mathrm{d}\boldsymbol{x}. \tag{5.3}$$

Solution. (a) Suppose u satisfies (5.1) and (5.2), then by direct calculation, we have

$$\frac{\partial^2 u}{\partial t^2} = \omega^2 \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \quad \Delta u = -|\boldsymbol{k}|^2 \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{5.4}$$

so we have (5.1) is equivalent to

$$(\omega^2 - |\boldsymbol{k}|^2 + m^2)\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = 0, \tag{5.5}$$

since $|\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}| = 1$, so we have

$$\boxed{\omega^2 = |\boldsymbol{k}|^2 - m^2}$$

is the formula we want to find.

(b) Define

$$\mathscr{E}'(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left[\left(\frac{\partial u}{\partial t}\right)^2 + |\nabla u|^2 + \frac{m^2}{2}u^2\right] \mathrm{d}\boldsymbol{x}, \tag{5.6}$$

and we assume u is compactly supported then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}'(u) = \int_{\mathbb{R}^n} \left[\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u\right)\right] \mathrm{d}\boldsymbol{x} = 0,$$

thus we know that the engery $\mathscr{E}'(u)$ is conserved and as desired. □

Remark 5.1. *In my solution, I assume u is compactly supported, because I cannot prove that* $\lim_{|\boldsymbol{x}|\to\infty} u(t, \boldsymbol{x}) = 0$, *so I just do the same thing as Prof. Hu has done in dealing with Schrödinger's equation.*

References

[1] D. Borthwick, *Introduction to partial differential equations.* Springer, 2017.