Sympletic Geometry seminar.

§ Reeb vector fields.

Let (M, H) be a contact Mfld. α is a contact form.

Recall: ① $H \subset TM$ is a codim 1 hyperplanes 

② $H_p = \ker \alpha_p$. $d\alpha|_H$ is symplectic

$\Longrightarrow$ ③ contact Mfld has odd dimension

④ $\alpha \wedge (d\alpha)^n$ is a volume form on M.

Now we want consider the dynamic system on M

$\longrightarrow$ a special vector field.

$\boxed{\text{Claim}}$: there $\exists!$ $R \in \mathfrak{X}(M)$. st.

$$\begin{cases} i_R d\alpha = 0 \\ i_R \alpha = 1 \end{cases} \Longleftrightarrow \begin{cases} d\alpha(R, \cdot) = 0 & (1) \\ \alpha(R) = 1 \end{cases}$$

(1) always holds solution $\ker d\alpha$ is a line bundle.

one can easily find a global section

(2) $\ker d\alpha \cap \ker \alpha = \{0\}$. (from pic)

$\Rightarrow$ we can normalize R.

(Def 1) we call R is the Reeb vector field determined by α.

$\boxed{\text{Prop 2}}$: the flow of R preserves the contact form.

i.e. $\rho_t = \exp tR$. then $\rho_t^*\alpha = \alpha$. $\forall t \in \mathbb{R}$

Proof: $\frac{d}{dt}\rho_t^*\alpha = \rho_t^* \mathcal{L}_R \alpha$ $\left(\frac{d}{ds}\Big|_{s=t}\rho_s^*\alpha = \frac{d}{ds}\Big|_{s=t}\rho_t^*\rho_{s-t}^*\alpha = \rho_t^*\frac{d}{d\ell}\Big|_{\ell=0}\rho_\ell^*\alpha = \rho_t^*\mathcal{L}_R\alpha \right)$

Recall cartan magic formula:

$$\mathcal{L}_R\alpha = d i_R\alpha + i_R d\alpha = 0.$$

$\Rightarrow \frac{d}{dt}\rho_t^*\alpha = 0 \Rightarrow \rho_t^*\alpha = \rho_0^*\alpha = \alpha$ □

$\boxed{\text{Def 3}}$ contactomorphism $f_*H = H \Leftrightarrow f^*\alpha = \alpha$.

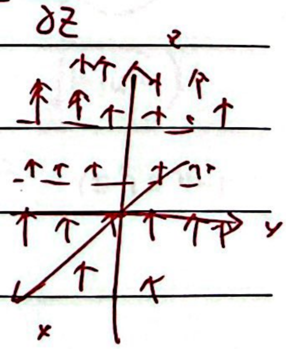
$\boxed{\text{Example 4}}$ $\mathbb{R}^{2n+1}$ with contact form $\alpha = \sum x^i dy^i + dz$.

(we have seen H in $\mathbb{R}^3$).

- determine the Reeb vector field.

$$\begin{cases} i_R\left(\sum dx^i \wedge dy^i\right) = 0 \\ \left(\sum x^i dy^i + dz\right)(R) = 1 \end{cases} \longrightarrow \text{observation: } R = \frac{\partial}{\partial z}$$

is a solution. and from uniqueness!



$\Rightarrow$ the contactomorphisms generated by R is

$$\rho_t(x^1 \dots y^n, z) = (x^1, y^1, \dots y^n, z+t).$$

$\boxed{\text{Examples}}$ consider $S^{2n-1} \xhookrightarrow{i} \mathbb{R}^{2n}$.

$\sigma \in \Omega^1(\mathbb{R}^{2n})$. $\sigma = \frac{1}{2}\sum (x^i dy^i - y^i dx^i)$

$\boxed{\text{claim:}}$ $\alpha = i^*\sigma$ is a contact form on S^{2n-1}.

Proof: only need to check:

$$\alpha \wedge (d\alpha)^{n-1} = i^*\alpha \wedge (d\, i^*\sigma)^{n-1} = i^*(\sigma \wedge (d\sigma)^{n-1}) \neq 0$$

- $d\sigma = \sum dx^i \wedge dy^i$

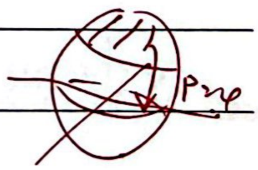
$$\Rightarrow (d\sigma)^{n-1} = \cancel{\Omega_n}\,(n-1)!\sum_j (dx^1\wedge dy^1)\wedge\cdots\wedge \widehat{dx^j\wedge dy^j}\wedge\cdots\wedge dx^n\wedge dy^n$$

$\Rightarrow \sigma \wedge (d\sigma)^{n-1} =$

$$\sum_j (x^j dy^j - y^j dx^j)\wedge (dx^1\wedge dy^1)\wedge\cdots\wedge \widehat{\cdots}\wedge\cdots\wedge dx^n\wedge dy^n$$

$\boxed{\text{step II}}$ • now we calculate i^*. we chose

$$\varphi: (x^1, y^1, \dots, x^n, y^n) = (x^1, y^1, \dots, x^n)$$



then: $i: (u^1, \dots, u^{2n-1}) \longmapsto (u^1, u^2, \dots, u^{2n-1}, \sqrt{1-|u|^2})$

$\Rightarrow i^* dx^j = du^{2j-1}$, $1 \le j \le n$. $i^* dy^j = du^{2j}$. $1 \le j \le n-1$

$dy^n \circ i = d(\sqrt{1-|u|^2})$.

$$\Rightarrow i^*(\sigma\wedge(d\sigma)^{n-1}) = \sum_{j=1}^{n-1}(u^{2j-1}du^{2j} - u^{2j}du^{2j-1})\wedge\widehat{(du^1\wedge du^2)}\wedge\cdots\wedge du^{2n-1}\wedge d(\sqrt{1-|u|^2})$$

$$+ \left(u^{2n-1}\, d\left(\sqrt{1-|u|^2}\right) - \overset{\sqrt{1-|u|^2}}{u^{2n}}\, du^{2n-1}\right)\wedge du^1\wedge\cdots\wedge du^{2n-2}$$

$$u^{2j-1}du^{2j} - u^{2j}du^{2j-1}$$

$$= \sum_{j=1}^{n-1} \cancel{\pm}(du^1\wedge du^2)\wedge\cdots\wedge\widehat{\ }\wedge du^{2n-1}\wedge\left(-\frac{u^{2j-1}du^{2j-1}}{\sqrt{1-|u|^2}} - \frac{u^{2j}du^{2j}}{\sqrt{1-|u|^2}}\right)\wedge(\ )$$

$$-\sqrt{1-|u|^2}\,dV - u^{2n-1}\cdot\frac{2u^{2n-1}}{\sqrt{1-|u|^2}}\,dV$$

$$= \sum_{j=1}^{n-1} \frac{-\cancel{2}\left((u^{2j-1})^2+(u^{2j})^2\right)}{\sqrt{1-|u|^2}}\,dV - \frac{1-|u|^2+2(u^{2n-1})^2}{\sqrt{1-|u|^2}}\,dV$$

$$= \cancel{\Rightarrow} -\frac{1-|u|^2+2|u|^2}{\sqrt{1-|u|^2}}\,dV = -\frac{1+|u|^2}{\sqrt{1-|u|^2}}\,dV \neq 0.$$

□

$\boxed{\text{Proof on the Book}}$: $\nu\wedge\sigma\wedge(d\sigma)^{n-1}\neq 0$

$$\Rightarrow \nu\wedge\sigma\wedge(d\sigma)^{n-1}(\underset{\text{normal vectors}}{n}, \underbrace{X_1,\dots,X_{2n-1}}_{\text{tangent vectors}}) = \underset{\neq 0}{\underline{\nu(n)}}\left(\sigma\wedge(d\sigma)^{n-1}\right)(X_1\cdots X_{2n-1})\neq 0$$

$$\Rightarrow \sigma\wedge(d\sigma)^{n-1}(X_1\cdots,X_{2n-1})\neq 0 \Rightarrow i^*\sigma\wedge i^*(d\sigma)^{n-1}\neq 0.$$

□

we know (S^{2n-1},σ) is a contact Mfld.

$\boxed{\text{step2}}$ ~~• Now determine the Reeb vector field:~~

$\alpha = i^*\sigma$. Find R: $\underline{i_*R} \to$ in $(x^1 \cdots y^n)$

$$\begin{cases} d\alpha(R,\cdot)=0 \longrightarrow d\sigma(i_*R,\cdot)=0 \\ \alpha(R)=1 \longrightarrow \sigma(i_*R)=1 \end{cases}$$

~~construct~~ $\underline{i_*R = 2\sum (x^i\partial_{y_i} - y^i\partial_{x^i})}$. $\begin{cases} \in T_pS^n. \ (\perp (x^1 \cdots y^n)) \\ \sigma(i_*R)=1 \\ d\sigma(i_*R,\cdot)=0. \end{cases}$

$$\sigma(i_*R) = \tfrac{1}{2}\left(\sum x^i dy^i - y^i dx^i\right)\left(2\sum x^i\partial_{y_i} - y^i\partial_{x^i}\right)$$

$$= \sum (x^i)^2 dy^i(\partial_{y^i}) + (y^i)^2 dx^i(\partial_{x^i}) = \sum (x^i)^2 + (y^i)^2 = 1$$

$d\sigma = \sum dx^i \wedge dy^i \qquad d\sigma(i_*R, X)$

$$= \sum dx^i\wedge dy^i(x^j\partial_{y^j} - y^j\partial_{x^j}, X)$$

$$= \sum_i \begin{vmatrix} dx^i(i_*R) & dx^i(X) \\ dy^i(i_*R) & dy^i(X) \end{vmatrix} = \sum_i \begin{vmatrix} -y^i & dx^i(X) \\ x^i & dy^i(X) \end{vmatrix}$$

$$= -\sum_i x^i dx^i(X) + y^i dy^i(X) = -\tfrac{1}{2} d\left(\sum (x^i)^2 + (y^i)^2\right)(X) = 0.$$

Finally: $\boxed{i_*R = 2\sum x^i\partial_{y_i} - y^i\partial_{x_i}}$

Step 3

More about Reeb vector field on S^{2n-1}

Hopf fibration:

$$\pi: S^{2n-1} \to \mathbb{C}P^{n-1}$$

S^{2n-1} can be viewed as a S^1 bundle over $\mathbb{C}P^{n-1}$

Proof: Show Reeb vector field gives the Hopf fibration of $S^{2n-1} = \{(z^1, \dots, z^n) \mid |z^1|^2 + \dots + |z^n|^2 = 1\}$

$\pi: S^{2n-1} \to \mathbb{C}P^{n-1} \quad (z^1, \dots, z^n) \mapsto [z^1 : \dots : z^n]$

$(z^1, \dots, z^n) \sim (x^1, y^1, \dots, x^n, y^n)$

suppose from $(x^1, y^1, \dots, x^n, y^n)$, $r(t) = (x^1(t), \dots, y^n(t))$

$\dot{r}(t) = (\dot{x}^1(t), \dots, \dot{y}^n(t)) = (-y^1(t), x^1(t), \dots)$ Reeb vector field

$$\begin{cases} \dot{x}^i(t) = -y^i(t) \\ \dot{y}^i(t) = x^i(t) \end{cases} \Rightarrow \ddot{x}^i(t) = x^i(t)$$

$\Rightarrow x^i(t) = A^i \cos t + B^i \sin t$

$\Rightarrow r(t) = (x^1 \cos t - y^1 \sin t, \ x^1 \sin t + y^1 \cos t, \dots)$

$\longrightarrow (z^1 e^{it}, z^2 e^{it}, \dots, z^n e^{it}) \xmapsto{\pi} [z^1 : \dots : z^n]$

Hopf fibration! $\square$

An interesting fact about Reeb

If M is a cmpt Mfld, $f \in C^\infty(M)$, with only two critical points, both of which are non-degenerate, then M is homeomorphic to S^n

$\hookrightarrow$ ① Morse theory
② not diffeomorphic (S^7 exotic sphere)

§2 Symplectization

contact topology $\longleftrightarrow$ symplectic topology. 先讲 Prop 6

~~First of all~~ Next let's see an example:

$\boxed{\text{Example 7}}$ (Symplectization of S^{2n-1}).

(Goal) From a contact Mfld to construct a symplectic manifold (dim +1 $\rightsquigarrow$ $\times\mathbb{R}$).

Consider $\pi: S^{2n-1} \times \mathbb{R} \cong \mathbb{R}^{2n} \setminus \{0\}$.

$(p, \tau) \longmapsto \sqrt{e^\tau}\, p$ (to avoid $\tau < 0$).

Goal: check $\left(S^{2n-1} \times \mathbb{R},\ d(e^\tau \pi^* \alpha)\right)$ is symplectomorphism to $(\mathbb{R}^{2n}, \Sigma dx^i$

Now: $\pi: \mathbb{R}^{2n}-\{0\} \xrightarrow{\pi} S^{2n-1}, \alpha \xhookrightarrow{i} \mathbb{R}^{2n}, \sigma$ 不写 x, y

$$(x^1, y^1, \cdots, x^n, y^n) \longmapsto \left(\frac{x^1}{\sqrt{e^\tau}}, \cdots, \frac{y^n}{\sqrt{e^\tau}}\right) \qquad (x^1, \cdots, y^n)$$

where $e^\tau = \sum (x^i)^2 + (y^i)^2$ consider the symplectic form on $\mathbb{R}^n - \{0\}$

$\boxed{\text{Construction}}$ $\omega = d(e^\tau \pi^* \alpha)$ is a symplectic form

$$\pi^*\alpha = \pi^* i^* \sigma = (i \circ \pi)^* \frac{1}{2}\left(\sum x^i dy^i - y^i dx^i\right)$$

$$= \frac{1}{2} \sum (x^j \circ i \circ \pi) d(y^j \circ i \circ \pi) - (y^j \circ i \circ \pi) d(x^j \circ i \circ \pi)$$

$$= \frac{1}{2} \sum \left(\frac{x^j}{\sqrt{e^\tau}}\right) d\left(\frac{y^j}{\sqrt{e^\tau}}\right) - \frac{y^j}{\sqrt{e^\tau}} d\left(\frac{x^j}{\sqrt{e^\tau}}\right) \qquad \left[\frac{1}{2e^\tau} \sum x^j dy^j - y^j dx^j + \underbrace{\frac{1}{2} \sum \frac{x^j y^j}{\sqrt{e^\tau}} d\left(\frac{1}{\sqrt{e^\tau}}\right) - \frac{y^j x^j}{\sqrt{e^\tau}} d\left(\frac{1}{\sqrt{e^\tau}}\right)}_{0}\right]$$

$$= \frac{1}{2e^\tau} \sum x^j dy^j - y^j dx^j$$

$\Rightarrow d\omega = \sum dx^i \wedge dy^i$ is the standard symplectic form.

$\square$.

Now we generalize to more general contact mfld.

$\boxed{\text{Prop 6}}$: Let (M, H) be a contact Mfld with contact form α. Let $\tilde{M} = M \times \mathbb{R}$.

$\pi: \tilde{M} \to M$. $(p, \tau) \longmapsto p$. then $\omega = d(e^\tau \pi^* \alpha)$ is a symplectic form on $\tilde{M}$ where τ is a coordinate on $\mathbb{R}$.

Proof: check: ① closed ✓

② non degenerate

$\Longleftrightarrow$ check $\omega^n \neq 0$.

$$\omega = e^{\tau} d\tau \wedge \pi^* \alpha + e^{\tau} d(\pi^* \alpha)$$

$$\Rightarrow \omega^n = \underbrace{(e^{\tau} d(\pi^*\alpha))^n}_{\overset{\parallel}{0}\ e^{\tau}\pi^*(d\alpha)^n} + \binom{n}{1} e^{\tau} d\tau \wedge \pi^* \alpha \wedge (e^{\tau} d(\pi^*\alpha))^{n-1}$$

since α at most $2n-1$ coordinate

$$= n e^{n\tau} d\tau \wedge \pi^*(\underline{\alpha \wedge (d\alpha)^{n-1}})$$

$\to$ form α is contact form $\neq 0$.

then we know ω is really a symplectic form. $\square$.

Rmk. we call $(\tilde{M}, \omega)$ is the symplectization of (M, α).

Let's see the symplectization of S^{2n-1}

§3 Contact dynamics

Main problem: we concern about the dynamic systems on three-Mfld. (Sphere)
↓ the flow generated by nowhere vanishing vector field.

Rmk 1: why 3-sphere. (2-sphere does not have nowhere vanishing vector field (moo Olu thm).

Question 1: (Seifert 1948) $v \in \mathcal{X}(S^3)$. $v \neq 0$.
Does the flow v have any periodic orbits?

Example 2: S^3, and its Reeb vector field
$\rightsquigarrow$ the orbits of flow are the circles of the Hopf fibration.

Result 3: Not True!

- C^1 counterexample Schweitzer 1974
- C^∞ counterexample K. Kuperberg 1994.

Annals

(平均的人. 佩服一候)

Question 4 ~~consider some special vector fields~~

i.e. preserve some structure, geometrically (volume-preserving).

Rmk: what is volume presering? suppose the flow generated by X is φ_t, $\exp tX$. then $(\exp tX)^*\omega = \omega$.
ω is the volume form. (exactly preserve volume).

Direct result 5:
Kuperberg 1997. $\exists C^1$ counterexample
C^∞ is not known for now.

Now we see how they related.
$\boxed{\text{Dynamic system} \leftrightarrow \text{contanct Geometry}}$

• Hypthesis: If $M = S^3$. r is a volume form

Goal: Find $v \in \mathfrak{X}(S^3)$. $v \neq 0$, v is volume-preserving
What "v" looks like?

$\because$ Volume preserving $\Rightarrow \mathcal{L}_v r = 0 \Rightarrow (di_v + i_v d) r = 0$

$\Rightarrow d(i_v \gamma) = 0.$ (note $H^2(S^3) = 0$)

$\Rightarrow \exists \alpha \in \Omega^1(S^3).\ i_v \gamma = d\alpha$

· Summar~~ization~~y: If v preserve volume

$\Rightarrow \exists$ 1-form α. $i_v \gamma = d\alpha$.

note that $i_v i_v \gamma = 0 \Rightarrow i_v d\alpha = 0$

Recall: Reeb vector field

$$\begin{cases} i_R d\alpha = 0 \\ i_R \alpha = 1 \\ \alpha \wedge d\alpha \text{ is volume form} \end{cases}$$

~~Thus summary~~ If on S^3, we can find a contact form α and Reeb vector field R. then we can study seifert conjecture.

Question 6 weinstein 1978 | Berkeley student of chern

Suppose that M is a 3-Mfld. with a contact form α. V be the Reeb vector field of $\alpha \Rightarrow V$ has a periodic orbit.

Result 7 (Viterbo & Hofer)

If 1) $M = S^3$

or 2) $\pi_2(M) \neq 0$

or 3) the contact structure is overtwisted

$\Rightarrow$ M admits a contact form α.

and v is the Reeb vector field of α

then v has a periodic orbit