Complex Geometry

Notes and Problems

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Preface

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1 Preliminaries

1.1 Notes

Definition 1.1.1: Holomorphic Function

Let $f = (f^1, \cdots, f^m) : \mathbb{C}^n \to \mathbb{C}^m$, then f is **holomorphic**, if

$$\boxed{\frac{\partial u^i}{\partial x^j} = \frac{\partial v^i}{\partial y^j}, \quad \frac{\partial u^i}{\partial y^j} = -\frac{\partial v^i}{\partial x^j},} \quad \forall 1 \le i \le m, 1 \le j \le n,$$

where $f^i = u^i + \sqrt{-1}v^i$, we can also denote it as $\overline{\partial} f = 0$.

Recall Cauchy integral formula for $\mathbb{C}$

Theorem 1.1.1

Suppose $\Omega \subseteq \mathbb{C}$ is a bounded domain, $\partial\Omega$ piecewise C^1 and is a Jordan curve, then let $f \in C^(\Omega)$, and for any $z_0 \in \Omega$, we have

$$f(z_0) = \frac{1}{2\pi\sqrt{-1}} \left(\int_{\partial\Omega} \frac{f(z)}{z - z_0} \mathrm{d}z + \iint_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}}(z)}{z - z_0} \mathrm{d}z \wedge \mathrm{d}\bar{z} \right),$$

then we have two special cases

1. If f is holomorphic, then we have

$$f(z_0) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega} \frac{f(z)}{z - z_0} \mathrm{d}z.$$

2. If $f \in C_0^1(\Omega)$ with compact support contained in Ω, then

$$f(z_0) = \frac{1}{2\pi\sqrt{-1}} \iint_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}}(z)}{z - z_0} \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Proof. One can transfer it to the real version then use Grenn formula. ♣

Now we generalize it to the multi-complex variable :

Theorem 1.1.2

Suppose $\Omega \subseteq \mathbb{C}^n$, let $f \in \mathcal{O}(\mathbb{C}^n)$, for any $\xi \in \Omega$, let the **polydisk** be

$$\mathbb{D}_{\boldsymbol{r}}(\xi) := \{(z^1, \cdots, z^n) \in \mathbb{C}^n : |z^i - \xi^i| < r^i\},$$

where $\xi=(\xi^1,\cdots,\xi^n)$ and $\boldsymbol{r}=(r^1,\cdots,r^n)$,then for $\mathbb{D}_{\boldsymbol{r}}(\xi)\subseteq\Omega$, we have

$$f(\xi)=\frac{1}{(2\pi\sqrt{-1})^n}\int_{|z^1-\xi^1|=r^1}\cdots\int_{|z^n-\xi^n|=r^n}\frac{f(z^1,\cdots,z^n)}{(z^1-\xi^1)\cdots(z^n-\xi^n)}\mathrm{d}z^1\cdots\mathrm{d}z^n.$$

Then compare to the single value function ,we have Taylor expansion:

Corollary 1.1.1

For any $f\in\mathcal{O}(\Omega)$, $\xi\in\Omega$, then there exists $\mathbb{D}_{\boldsymbol{r}}(\xi)\subseteq\Omega$, such that

$$f(z^1,\cdots,z^n)=\sum_{\alpha\in\mathbb{N}^n}C_\alpha(z-\xi)^\alpha,\quad (z-\xi)^\alpha=\prod_{i=1}^n(z^i-\xi^i)^{\alpha_i},$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)$ and we also have

$$C_\alpha=\frac{1}{(2\pi\sqrt{-1})^n}\int_{|z^1-\xi^1|=r^1}\cdots\int_{|z^n-\xi^n|=r^n}\frac{f(z^1,\cdots,z^n)}{(z^1-\xi^1)^{\alpha_1+1}\cdots(z^n-\xi^n)^{\alpha_n+1}}\mathrm{d}z^1\cdots\mathrm{d}z^n.$$

More precisely, we have

$$C_\alpha=\frac{1}{(\alpha_1!)\cdots(\alpha_n!)}\cdot\frac{\partial^{\alpha_1+\cdots\alpha_n}f}{\partial(z^1)^{\alpha_1}\cdots\partial(z^n)^{\alpha_n}}.$$

Now we see something different

Theorem 1.1.3: Hartogs Extension

Let $\Omega\subset\mathbb{C}^n$ be a domain, and $n\geq 2$, $K\subseteq\Omega$ is compact, and $\Omega\setminus K$ is connected, let $f\in\mathcal{O}(\Omega\setminus K)$, then there exists $\widetilde{f}\in\mathcal{O}(\Omega)$ such that $\widetilde{f}=f$ on $\Omega\setminus K$.

Remark. when $n=1$, the theorem dosen't hold, since $f(z)=\dfrac{1}{z}$ on $\mathbb{C}-\{0\}$ is a counterexample.

Proof. The main idea is to solve $\bar\partial$ equation.

Firstly, we use cut off function to smoothly extend f, suppose $K\subset U_1\subset U_2\subset\Omega$, then let $\varphi\in C^\infty(\mathbb{C}^n)$, and $\varphi\equiv 0$ on U_1, and $\varphi\equiv 1$ on $\mathbb{C}^n\setminus U_2$, now we have $\varphi f\in C^\infty(\Omega,\mathbb{C})$.

Goal : Find $g\in C_0^\infty(\Omega,\mathbb{C})$, such that $\overline{\partial}(g-\varphi f)=0$, beacuse given such g, let $\widetilde{f}=\varphi f-g$, we know that $\widetilde{f}$ is holomorphic, and it equals to f on the boundary of Ω, then from the uniqueness of holomorphic functions, note $\Omega\setminus K$ is connected, so $f=\widetilde{f}$.

So we now construct g, let $u_i=\dfrac{\partial\varphi}{\partial\bar z^i}\cdot f=\dfrac{\partial}{\partial\bar z^i}(\varphi f)\in C_0^\infty(\mathbb{C}^n,\mathbb{C})$, then we have $\dfrac{\partial u_i}{\partial\bar z^k}=\dfrac{\partial u_k}{\partial\bar z^i}$, for all i,k, then we can let

$$g(z^1,\cdots,z^n):=\iint_{\mathbb{C}}\frac{u_1(\tau,z^2,\cdots,z^n)}{\tau-z^1}\mathrm{d}\tau\wedge\mathrm{d}\bar\tau=\iint_{\mathbb{C}}\frac{u_1(\tau+z^1,z^2,\cdots,z^n)}{\tau}\mathrm{d}\tau\wedge\mathrm{d}\bar\tau,$$

Now to check our goal, it suffices to verify $\dfrac{\partial g}{\partial\bar z^k}=u_k$ and g has compact support, we omit them. ♣

Definition 1.1.2: Meromorphic Function

A function f on Ω is called **meromorphic** if there exists open cover $\mathcal{U} = \{U_i\}$ of Ω, such that

$$f = \frac{h_i}{g_i}, \quad h_i, g_i \in \mathcal{O}(U_i),$$

and we denote the all meromorphic functions to be $\mathcal{M}(\Omega)$.

2

Complex Manifolds

2.1 Complex Manifolds

Definition 2.1.1

A **complex manifold** M is a differentiable manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that

- (C1) $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism;
- (C2) whenever $U_\alpha \cap U_\beta \neq \varnothing$, we have $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a biholomorphic map from $\varphi_\beta(U_\alpha \cap U_\beta)$ to $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Remark. From Daniel we know that biholomorphic is equivalent to bijective and holomorphic.

Definition 2.1.2

1. A map $f : M \to \mathbb{C}$ from a complex manifold is called a **holomorphic function**, if $f \circ \varphi_i^{-1} \in \mathcal{O}(\varphi_i(U_i))$, for all $i \in I$, in this case, we write $f \in \mathcal{O}(M)$;
2. If M, N are both complex manifolds of dimension n and m respectively, a map $F : M \to N$ is called **holomorphic** if for all coordinate charts (U, φ) of M and (V, ψ) of N , the map $\psi \circ F \circ \varphi^{-1}$ is a holomorphic map on $\varphi(U \cap F^{-1}(V)) \subseteq \mathbb{C}^n$ whenever $U \cap F^{-1}(V) \neq \varnothing$. A holomorphic map with a holomorphic inverse is called **biholomorphic**.

Remark. We will define "$\mathcal{O}$" the holomorphic function sheaf later.

Example 2.1.1 (Some special complex manifolds)**.**

1. *Open subsets of $\mathbb{C}^n$ are complex manifolds;*
2. *Let $\{e_1, \cdots, e_{2n}\}$ be any fixed $\mathbb{R}-$basis of $\mathbb{C}^n$, and let*

$$\Gamma := \{m_1e_1 + \cdots + m_{2n}e_{2n} | m_i \in \mathbb{Z}\}$$

*be a lattice of rank $2n$. Then we can define the quotient space $\mathbb{C}^n/\Gamma$, it is a compact Haisdorff space equipped with quotient topology. There is a natural complex manifold structure induced from the quotient map on $\mathbb{C}^n/\Gamma$, we call this complex mainfold a **complex torus**;*

3. *Let $P \in \mathbb{C}[z, w]$ be a polynomial of degree d. Define*

$$C := \{(z, w)|P(z, w) = 0\}.$$

We call it an ***affine plane algebraic curve****, now assume P is irreducible and $\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}$ have no common zeros on C, i.e., ∇P nowhere vanishes, and then C is a natural complex manifold, more precisely, a non-compact Riemann surface.*

Proof. The coordinates can be choosen in the following way: WLOG if $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$, then we can apply the holomorphic version of implicit function theorem to find a neighborhood $B := B(z_0, \varepsilon) \times B(w_0, \delta)$, and a holomorphic function $g(z)$ such that $U := C \cap B = \{(z, w)|z \in B(z_0, \varepsilon), w = g(z)\}$, we choose $\varphi : U \to \mathbb{C}$ to be $\varphi(z, w) = z$, since $w = g(z)$ so it is really a homeomorphism, and furthermore, if $\frac{\partial P}{\partial z}(z_0, w_0) \neq 0$, we use w as local coordinate.

Now we consider the map on the intersection, if $(z_0, w_0) \in U_i \cap U_j$ with coordinate $\varphi_i(z, w) = z$, $\varphi_j(z, w) = w$, then suppose from implicit theorem we have $z = g(w)$ and $w = h(z)$ on $U_i \cap U_j$ then $\varphi_i \circ \varphi_j^{-1}(w) = g(w)$, since $g \circ h$ and $h \circ g$ both identity, so g is bijective and holomorphic then biholomorphic. ♣

More about Complex Tori

Let us consider the one-dimensional a bit more in detail. Suppose $\omega_1, \omega_2 \in \mathbb{C}^*$, and $\omega_1/\omega_2 \notin \mathbb{R}$, we denote Γ to be the discrete subgroup of $\mathbb{C}$ generated by ω_1 and ω_2,

$$\boxed{\Gamma := \omega_1\mathbb{Z} + \omega_2\mathbb{Z} = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}}.$$

So we have $\mathbb{C}/\Gamma = \mathbb{C}/\sim$, where $z \sim w$ if and only $\exists m, n \in \mathbb{Z}$ such that $z = w + m\omega_1 + n\omega_2$. We use $[z]$ to represent the equivalence class of z, and π is the quotient map, now we show in detailed that $\mathbb{C}/\Gamma$ is a Riemann surface, i.e., a 1-complex manifold:

Firstly, we denote

$$\delta = \inf_{(m,n)\neq(0,0)} |m\omega_1 + n\omega_2| > 0,$$

then for arbitrary $p \in \mathbb{C}/\Gamma$, let $z_p \in \pi^{-1}(p)$, and

$$W_p = \{w \in \mathbb{C} | |w - z_p| < \delta/2\}, \quad U_p = \pi(W_p),$$

so by the definition of quotient map, and $\pi|_{W_p}$ is a homeomorphism, U_p is open in $\mathbb{C}/\Gamma$, so let

$$\begin{aligned} \varphi_p : U_p &\to W_p \subset \mathbb{C} \\ p &\mapsto (\pi|_{W_p})^{-1}(p), \end{aligned}$$

so we have $\{(U_p, \varphi_p)\}$ is an atlas, since $\varphi_p \circ \varphi_q^{-1}(z) = z + \omega$ for some $\omega \in \Gamma$ is holomorphic, then we know that $\mathbb{C}/\Gamma$ is really a Riemann surface.

Now a natural question is *Are those complex tori holomorphic isomorphic ? If not, can we give them a classification ?* Before we answer these questions, we offer some basic propositions:

Proposition 2.1.1

1. the quotient projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a holomorphic covering map ;
2. For arbitrary two tori $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$, they are diffeomorphism;
3. Suppose $f : \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2$ is a continuous(holomorphic) map, and $f([0]) = [0]$, then there exists unique continuous(holomorphic) map $\widetilde{f} : \mathbb{C} \to \mathbb{C}$, such that $f(0) = 0$, and satisfies the following diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\widetilde{f}} & \mathbb{C} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{C}/\Gamma_1 & \xrightarrow{f} & \mathbb{C}/\Gamma_2 \end{array}$$

Proof. For 3, one can first lift f to a map from $\mathbb{C}/\Gamma_1$ to $\mathbb{C}$, then define $\widetilde{f}$. ♣

Theorem 2.1.1

1. For arbitrary $p \in \mathbb{C}/\Gamma$, there exists a holomorphic automorphism such that

$$f_p : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma, \quad f_p(p) = [0];$$

2. Complex tori $\mathbb{C}/(\omega_1, \omega_2)$, $\mathbb{C}/(1, \omega_1/\omega_2)$ and $\mathbb{C}/(1, \omega_2/\omega_1)$ are isomorphic.

Proof. (1) Fixed $z_p \in \pi^{-1}(p)$, define $f_p : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma$, $f_p([w]) := [w - z_p]$, then $f_p(p) = f_p([z_p]) = [0]$, and it is not hard to check that f_p is really a holomorphic automorphism.

(2) naturally condider $f : \mathbb{C}/(\omega_1, \omega_2) \to \mathbb{C}/(1, \omega_1/\omega_2)$, $f([z]) := [z/\omega_2]$. ♣

So from the theorem above, when we classify the complex tori, it sufficies to consider a special case, i.e., $\Gamma = (1, \tau)$ and $\mathrm{Im}\tau > 0$. Suppose $f : \mathbb{C}/(1, \tau) \to \mathbb{C}/(1, \tau')$ is a biholomorphic, then from the proposition above, we assume $f([0]) = [0]$, and $F : \mathbb{C} \to \mathbb{C}$ is the lift of f, and then F is also holomorphic, and satisfies $F(0) = 0$, $\pi' \circ F = f \circ \pi$.

Now similarly, suppose G is the lift of f^{-1}, so $F \circ G$ and $G \circ F$ are all the lift of identity, then from the uniqueness of lifting, we know that $F \circ G = \mathrm{id}$, $G \circ F = \mathrm{id}$. So F and G are the biholomorphism from $\mathbb{C}$ to $\mathbb{C}$. Recall a improtant and basic result :

Theorem 2.1.2: Aut($\mathbb{C}$)

If $F: \mathbb{C} \to \mathbb{C}$ is biholomorphism, then F is linear, i.e., $F(z) = az + b$ for some $a, b \in \mathbb{C}$.

So since $F(0) = 0$, there exists $\gamma \neq 0$, and $F(z) = \gamma \cdot z$, then we know that the biholomorphic $f: \mathbb{C}/(1,\tau) \to \mathbb{C}/(1,\tau')$ such as

$$f([z]) = [\gamma z], \quad \forall z \in \mathbb{C},$$

especially, we have

$$\begin{aligned} [0] &= f([0]) = f([1]) = [\gamma], \\ [0] &= f([0]) = f([\tau]) = [\gamma \cdot \tau], \end{aligned}$$

which implies that there exists $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{cases} \gamma = a \cdot 1 + b \cdot \tau', \\ \gamma \cdot \tau = c \cdot 1 + d \cdot \tau'. \end{cases}$$

Now we write the formula above in the matrix form

$$\gamma \cdot \begin{pmatrix} 1 \\ \tau \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau' \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}.$$

Simarly, we consider f^{-1}, then we obtain

$$\gamma^{-1} \cdot \begin{pmatrix} 1 \\ \tau' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix}, \quad a', b', c', d' \in \mathbb{Z}.$$

So we actually have

$$\begin{pmatrix} 1 \\ \tau \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \boldsymbol{I}_2.$$

where it comes from $1, \tau$ is linearly independent. Since a, b, c, d and a', b', c', d' are all integers, then we have $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$, and since $\mathrm{Im}\tau > 0$ and $\mathrm{Im}\tau' > 0$, so it is not hard to find $ad - bc = 1$. In short, we have the following classification theorem:

Theorem 2.1.3

If $\mathbb{C}/(1,\tau)$ and $\mathbb{C}/(1,\tau')$ are complex tori with $\mathrm{Im}\tau > 0$ and $\mathrm{Im}\tau' > 0$, then they are biholomorphism if and only if there exists $a, b, c, d \in \mathbb{Z}$ such that $\tau' = \dfrac{a + b\tau}{c + d\tau}$ with $ad - bc = 1$.

The Complex Projective Space

Definition 2.1.3

Let $\mathbb{CP}^n$ denote the set of lines through the origin in $\mathbb{C}^{n+1}$, a line $l \in \mathbb{C}^{n+1}$ is determined by any $\boldsymbol{z} \neq 0 \in l$, so we can write

$$\mathbb{CP}^n = \{[\boldsymbol{z}] \neq 0 \in \mathbb{C}^{n+1}\} \Big/ \boldsymbol{z} \sim \lambda \boldsymbol{z}, \quad \forall \lambda \in \mathbb{C}.$$

More precisely, we define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$: $(z_0, \cdots, z_n) \sim (w_0, \cdots, w_n)$ iff $\exists \lambda \in \mathbb{C}^*$ such that $w_i = \lambda z_i$, for all i. The $n-$dimensional **complex projective space** $\mathbb{CP}^n$ is defined to be the space with quotient topology, it is compact and Hausdorff since it can be viewed as a quotient space of sphere.

Now we choose holomorphic coordinate charts as follows: define $U_i := \{[z_0, \cdots, z_n] \in \mathbb{CP}^n | z_i \neq 0\}$, and define

$$\varphi_i : U_i \to \mathbb{C}^n, \quad \varphi_i([z_0, \cdots, z_n]) := \left(\frac{z_0}{z_i}, \cdots, \frac{\widehat{z_i}}{z_i}, \cdots, \frac{z_n}{z_i}\right) \in \mathbb{C}^n.$$

It is trivially a homeomorphism, now we check the compatibility, on $U_i \cap U_j$ and $\varphi_j(U_i \cap U_j) = \{(w_1, \cdots, w_n) \in \mathbb{C}^n | w_i \neq 0\}$, so we have

$$\varphi_i \circ \varphi_j^{-1}(w_1, \cdots, w_n) = \varphi([w_1, \cdots, 1, \cdots, w_n]) = \left(\frac{w_0}{w_i}, \cdots, \frac{\widehat{w_i}}{w_i}, \cdots, \frac{1}{w_i}, \cdots, \frac{w_n}{w_i}\right).$$

Remark. It is easy to check that $\mathbb{CP}^1$ is diffeomorphic to S^2.

Definition 2.1.4: Complex Submanifolds

A closed subset N of a $n-$dimensional complex manifold M is called a (closed) **complex submanifold** of dimension k, if for any $p \in N$, we can find a compatible chart (U, φ) of M such that $p \in U$ and $\boxed{\varphi(U \cap Y) = \{(z_1, \cdots, z_n) \in \varphi(U) \subseteq \mathbb{C}^n | z_{k+1} = \cdots = z_n = 0\}}$, one can check that the restriction of such charts (we call them "adapted charts") to N makes N a complex manifold and the inclusion $N \subseteq M$ is a holomorphic map.

Theorem 2.1.4

Any holomorphic function on a compact connected complex manifold should be a constant.

Proof. Recall **Maximum Principle** in chapter 1, i.e, let $\Omega \subseteq \mathbb{C}^n$ be a domain, and if $f \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$, then $\max_{\Omega} |f|$ can not be achieved at an interior point unless f is a constant.

Since M is compact, thus for the holomorphic function $f : M \to \mathbb{C}$, there exists $K \geq 0$, $\max_{M} |f| = K$, suppose $M_1 = \{p \in M | |f(p)| = K\}$, thus M_1 is closed in M. Furthermore, for all $p \in M_1$, consider a coordinate neighborhood (U, φ) of p, then $|f \circ \varphi^{-1}|$ attains its maximum in $\varphi^{-1}(U)$, thus from maximum principle, we knows that $f(U) = \{f(p)\}$, thus M_1 is open in M. Note M is connected, and M_1 is an open and closed subset of M, then we know that $M_1 = M$, i.e. f is a constant. ♣

From the theorem above, we have

Corollary 2.1.1

There are no compact complex submanifolds of $\mathbb{C}^n$ of positive dimension.

Proof. suppose $\dim_{\mathbb{C}} M = k > 0$, and M is a compact complex submanifols of $\mathbb{C}^n$, then forall $p = (z_1, \cdots, z_n) \in M \subseteq \mathbb{C}^n$, and since the inclusion map $i : M \hookrightarrow \mathbb{C}^n$ is holomorphic, then $i_k : M \to \mathbb{C}$, and $p \mapsto z_k$ is a holomorphic function, now from the theorem above, we know that i_k is a constant, then $i(M)$ is a point in $\mathbb{C}^n$, with dimension 0, a contradiction. ♣

Remark. This corollary means that we can not hope there is something like Whitney embedding theorem such that every complex manifold can be viewed as a submanifold of $\mathbb{C}^N$, where N is sufficient large. But fortunately, the complex projective space $\mathbb{CP}^n$ can be the new target space to be embedded in.

Those non-compact complex manifolds with admit proper holomorphic embeddings into $\mathbb{C}^N$ for some large N are precisely called **Stein manifolds**.

Projective Algebraic Manifolds

Definition 2.1.5

Let $F_1, \cdots, F_k \in \mathbb{C}[z_0, \cdots, z_n]$ be a set of irreducible homogeneous polynomials of degrees $d_1, \cdots, d_k$ respectively, then the set

$$\begin{aligned} V(F_1, \cdots, F_k) :&= \{\boldsymbol{z} = (z_0, \cdots, z_n) | F_1(\boldsymbol{z}) = \cdots = F_k(\boldsymbol{z}) = 0\} \Big/ \boldsymbol{z} \sim \lambda \boldsymbol{z} \\ &= \{[\boldsymbol{z}] = [z_0, \cdots, z_n] | F_1(\boldsymbol{z}) = \cdots = F_k(\boldsymbol{z}) = 0\} \end{aligned}$$

is well defined and is called a **complex projective algebraic variety**.

If we assume that $V(F_1, \cdots, F_k)$ is a complex submainfold of $\mathbb{CP}^n$, then it will be called a **projective algebraic manifold** or **Hodge manifold**.

Remark. Note that homogenous means that $F_i(\lambda \boldsymbol{z}) = \lambda^{d_i} F(\boldsymbol{z})$ for all $1 \leq i \leq k$.

Example 2.1.2. *Let F be irreducible and homogeneous of degree d. If the only common zero of $\dfrac{\partial F}{\partial z_0}, \cdots, \dfrac{\partial F}{\partial z_n}$ in $\mathbb{C}^{n+1}$ is $(0, \cdots, 0)$, then $V(F)$ is a complex submanifold of dimension $n-1$.*

*For example, $V(z_0^d + \cdots + z_n^d)$ is a smmoth submanifold of $\mathbb{CP}^n$, called the **Fermat hypersurface** of degree d.*

Proof. We check this on $U_0 = \{z_0 \neq 0\}$, then $V(F) \cap U_0$ is the zero locus of the holomorphic function $F(1, z_1, \cdots, z_n) \in \mathcal{O}(U_0)$, so from the implicit function theorem, it suffices to show that $\frac{\partial F}{\partial z_1}(1, \boldsymbol{z})$, $\cdots$, $\frac{\partial F}{\partial z_n}(1, \boldsymbol{z})$ have no common zeros on $V(F) \cap U_0$, where $\boldsymbol{z} = (z_1, \cdots, z_n)$.

We argue it by contracdiction, if not, then there exists $\boldsymbol{z}_0 = (z_1^0, \cdots, z_n^0)$ and

$$F(1, z_1^0, \cdots, z_n^0) = \frac{\partial F}{\partial z_1}(1, \boldsymbol{z}_0) = \cdots \frac{\partial F}{\partial z_n}(1, \boldsymbol{z}_0) = 0.$$

By **Euler's theorem** on homogeneous functions, we have

$$1 \cdot \frac{\partial F}{\partial z_0}(1, \boldsymbol{z}_0) + z_1^0 \frac{\partial F}{\partial z_1}(1, \boldsymbol{z}_0) + \cdots + z_n^0 \frac{\partial F}{\partial z_n}(1, \boldsymbol{z}_0) = d \cdot F(1, \boldsymbol{z}_0) = 0,$$

This implies that $\frac{\partial F}{\partial z_0}(1, \boldsymbol{z}_0) = 0$, then $(1, \boldsymbol{z}_0)$ is a common zero of ∇F in $\mathbb{C}^{n+1}$ different from $(0, \cdots, 0)$, which is a contradiction. ♣

A generalization of submanifold is the following

Definition 2.1.6

A closed subset A of a complex manifold M is called an **analytic subvariety**, if it is locally the common zeros of finitely many holomorphic functions, i.e. for all $p \in A$, there is an open set $U \subseteq M$ and $f_1, \cdots, f_k \in \mathcal{O}(U)$ such that

$$A \cap U = \{z \in U | f_1(z) = \cdots = f_k(z) = 0\}.$$

An analytic subvariety A is called a **hypersurface** if it is locally the zero locus of **a** holomorphic function. For example, the Fermat hypersurface.

Now we talk about the relation between submanifolds and subvarieties:

- A complex submanifold is always an analytic subvariety, since we can just chhoose U to be the domain of the adapted chart and f_i to be $z_{k+1}, \cdots, z_n$.

- An analytic subvariety may locally be a complex submanifold, let $A \subseteq M$ be an analytic subvariety, $p \in A$ is called a **regular point**, if we can find open $U \subseteq X$ and $f_1, \cdots, f_k \in \mathcal{O}(U)$ such that $A \cap U = \{z \in U | f_1(z) = \cdots = f_k(z) = 0\}$ and rank $\frac{\partial(f_1, \cdots, f_k)}{\partial(z_1, \cdots, z_n)}(p) = k$. In this case, A is locally near p a complex submanifold of dimension $n - k$.

The locus of regular points of A is denoted by A_{reg}, its complement in A is called the **singular locus**, and its elements are called **singular points** of A.

And there is an amazing theorem about analytic variety and algebraic variety:

Theorem 2.1.5: Chow's Theorem

Complex analytic subvarieties of $\mathbb{CP}^n$ are algebraic.

Roughly speaking, when one see the words below in the left column, the words in the right column should appear in one's mind.

$$\boxed{\text{Variety}} \longleftrightarrow \text{Zero Locus}$$
$$\boxed{\text{Algebraic}} \longleftrightarrow \text{Homogeneous Polynomial}$$
$$\boxed{\text{Analytic}} \longleftrightarrow \text{Holomorphic Function}$$

Existence of Complex Structures On a Given C^∞ Manifold

Proposition 2.1.2

A complex manifold is an even dimensional orientable differential manifold.

Proof. Recall the definition of an orientable real manifold, i.e. there exists an atlas $\{U_i, \varphi_i\}$ such that whenever $U_i \cap U_j \neq \varnothing$, then $\det J_{\varphi_i \circ \varphi_j^{-1}} > 0$, since for complex manifold M the transition map $\varphi_i \circ \varphi_j^{-1}$ is holomorphic in $\mathbb{C}^n$, then it suffices to show that for holomorphic f, $\det J_f^{\mathbb{R}} > 0$.

Actually, we will show that $\det J_f^{\mathbb{R}} = |\det J_f^{\mathbb{C}}|^2$, suppose $z = X + \sqrt{-1}Y$ and $f = U + \sqrt{-1}V$, then from Cauchy-Riemann equation, we have

$$\frac{\partial U}{\partial X} = \frac{\partial V}{Y}, \quad \frac{\partial U}{\partial Y} = -\frac{\partial V}{\partial X}.$$

Recall $J_f^{\mathbb{C}} = \left(\frac{\partial f^i}{\partial z^j}\right)_{i,j}$, and actually using $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right)$, thus

$$\begin{aligned}\frac{\partial f^i}{\partial z^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}\right)(u^i + \sqrt{-1}v^i)\\ &= \frac{\partial u_i}{\partial x^j} - \sqrt{-1}\frac{\partial u^i}{\partial y^j},\end{aligned}$$

where the last equation we use the Cauchy-Riemann equation, so we have

$$J_f^{\mathbb{C}} = \frac{\partial U}{\partial X} - \sqrt{-1}\frac{\partial U}{\partial Y},$$

hence we know that

$$\begin{aligned}\det J_f^{\mathbb{R}} &= \det\begin{pmatrix}\frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y}\\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y}\end{pmatrix} = \det\begin{pmatrix}\frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y}\\ -\frac{\partial U}{\partial Y} & \frac{\partial U}{\partial X}\end{pmatrix}\\ &= \det\begin{pmatrix}J_f^{\mathbb{C}} & \frac{\partial U}{\partial Y}\\ -\sqrt{-1}J_f^{\mathbb{C}} & \frac{\partial V}{\partial Y}\end{pmatrix} = \det\begin{pmatrix}J_f^{\mathbb{C}} & \frac{\partial U}{\partial Y}\\ 0 & J_f^{\mathbb{C}}\end{pmatrix}\\ &= |\det J_f^{\mathbb{C}}|^2.\end{aligned}$$

Finally we finish the proof. ♣

Example 2.1.3. $\mathbb{RP}^{2n}$ *can never be viewed as a complex manifold.*

However, for a given even dimensional oriented manifold, it is not always clear whether or not we can make it a complex manifold.

But there are topological obstructions to **almost complex structure**, this can rule out all even dimension spheres except S^2 and S^6, we already knew S^2 is a complex manifold, but the S^6 case is still open, more details can be found here .

Calabi–Eckmann Manifolds

Theorem 2.1.6

We can make $S^{2p+1} \times S^{2q+1}$ into a complex manifold.

The idea is that we can write

$$S^{2p+1} = \left\{ z \in \mathbb{C}^{p+1} \middle| \sum_{i=0}^{p} |z_i|^2 = 1 \right\}, \quad S^{2q+1} = \left\{ z \in \mathbb{C}^{p+1} \middle| \sum_{i=0}^{q} |z_i|^2 = 1 \right\},$$

and we have the **Hopf fibration** maps :

$$\pi_p : S^{2p+1} \to \mathbb{CP}^n, \quad \pi_q : S^{2q+1} \to \mathbb{CP}^q,$$

each with fiber S^1, i.e., S^1 bundle, so if we consider the map

$$\pi = (\pi_p, \pi_q) : S^{2p+1} \times S^{2q+1} \to \mathbb{CP}^p \times \mathbb{CP}^q,$$

then we can view $\boxed{S^{2p+1} \times S^{2q+1} \text{ as a fiber bundle on } \mathbb{CP}^p \times \mathbb{CP}^q}$, which is a complex manifold (Note that the product of complex manifold is still a complex manifold), with fiber $S^1 \times S^1 = T^2$.

Since the base space and the fiber can all be viewed as complex manifolds, so a natural idea is that we may construct the complex chart using their charts. To be precise, we first consider Tours, fix a $\tau \in \mathbb{C}$ with $\mathrm{Im}\tau > 0$. We denote by $T^2 = T_\tau$ the complex torus $\mathbb{C}/\langle 1, \tau \rangle$.

Consider the open sets

$$U_{kj} := \{(z, z') \in S^{2p+1} \times S^{2q+1} | z_k z'_j \neq 0\},$$

and using the coordinate map of $\mathbb{CP}^p \times \mathbb{CP}^q$, we have the map

$$h_{kj} : U_{kj} \to \mathbb{CP}^p \times \mathbb{CP}^q \times T_\tau \xrightarrow{(\varphi_k, \psi_j)} \mathbb{C}^{p+q} \times T_\tau$$

$$\boxed{h_{kj}(z, z') = \left(\frac{z_0}{z_k}, \cdots, \widehat{\frac{z_k}{z_k}}, \cdots, \frac{z_p}{z_k}, \frac{z'_0}{z'_j}, \cdots, \widehat{\frac{z'_j}{z'_j}}, \cdots, \frac{z'_q}{z'_j}, t_{kj} \right)},$$

where $t_{kj}(z, z') := \frac{1}{2\pi\sqrt{-1}}(\log z_k + \tau \log z'_j) \bmod \langle 1, \tau \rangle$, note here "mod" to make log to be a single-valued function.

Then using the coordinate map from T_τ to $\mathbb{R}^2$ then one can get the final coordinate map of $S^{2p+1} \times S^{2q+1}$. The compatibility may be trivial to verify, but I don't know either.

Corollary 2.1.2: Kai Zhu

We can make $\mathbb{RP}^{2p+1} \times \mathbb{RP}^{2q+1}$ into a complex manifold.

Proof. I guess it is right, but I'm not sure. ♣

2.2 Vector Bundles

Holomorphic Vector Bundle

Roughly speaking, a holomorphic vector bundle over a complex manifold is a family of vector spaces, varying holomorphically.

Definition 2.2.1

A **holomorphic vector bundle** of rank r over a $n-$dimensional complex manifold M is a complex manifold E of dimension $n+r$, together with a holomorphic surjective map $\pi : E \to M$ satisfying:

1. (*Fiberwise Linear*) Each fiber $E_p := \pi^{-1}(p)$ has the structure of $r-$dimensional vector space over $\mathbb{C}$;
2. (*Locally Trivial*) Thereis an open cover of M, $\mathcal{U} = \{U_i\}_{i\in I}$ such that each $\pi^{-1}(U)_i$ is biholomorphic to $U_i \times \mathbb{C}^r$ via $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$, and $E_p \hookrightarrow \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ is a linear isomorphism onto $\{p\} \times \mathbb{C}^r$ for any $p \in U_i$. φ_i is called a **local trivialization**.

A vector bundle of rank 1 is usually called a **line bundle**.

Whenever $U_i \cap U_j \neq \varnothing$, we have a holomorphic map, called the **transition map**, $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$ (viewed as an open subset of $\mathbb{C}^{r^2}$) such that

$$\varphi_i \circ \varphi_j^{-1}(p, v) = (p, \psi_{ij}(p)v), \quad p \in U_i \cap U_j, \quad v \in E_p.$$

Those families of transition maps satisfies the **cocycle condition**:

1. $\psi_{ij}\psi_{ji} = \boldsymbol{I}_r$ on $U_i \cap U_j$, i.e. $\boldsymbol{I}_r(p) = \boldsymbol{I}_r$;
2. whenever $U_i \cap U_j \cap U_k \neq \varnothing$, we have $\boxed{\psi_{ij}\psi_{jk}\psi_{ki} = \boldsymbol{I}_r}$ on $U_i \cap U_j \cap U_k$.

Remark. The name "cocycle" is no coincidence. In fact we will see later that $\{\psi_{ij}\}$ above is indeed a cocycle in Čech's approach to sheaf cohomology theory. More precisely, we have

$$(\delta\psi)_{ijk} = \psi_{jk} \circ (\psi_{ik})^{-1} \circ \psi_{ij} = \psi_{jk}\psi_{ki}\psi_{ij} = \boldsymbol{I}_r \quad \Longleftrightarrow \quad \delta\psi = 0.$$

On the other hand, if we are given a set of holomorphic transition maps $\{\psi_{ij}\} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$ satisfying the cocycle condition, we can construct a holomorphic vector bundle by setting

$$E = \bigsqcup_{i\in I}(U_i \times \mathbb{C}^r) \Big/ \sim,$$

where $(p, v) \sim (q, w)$ for $(p, v) \in U_i \times \mathbb{C}^r$ and $(q, w) \in U_j \times \mathbb{C}^r$ iff $\boxed{p = q \text{ and } v = \psi_{ij}(p)w}$.

Definition 2.2.2

Let $\pi : E \to M$ be a holomorphic vector bundle over M. Let $U \subseteq M$ be an open set. A **holomorphic section** of E over U is a holomorphic map $s : U \to E$ such that $\boxed{\pi \circ s = \mathrm{id}_U}$, i.e., $s(p) \in E(p)$ for any $p \in U$. The set of holomorphic sections over U is usually denoted by $\Gamma(U, \mathcal{O}(E))$ or $\mathcal{O}(E)(U)$. (Note that the notation comes from sheaf theory)

Remark. Roughly speaking, you can really view a section as a $E-$valued function.

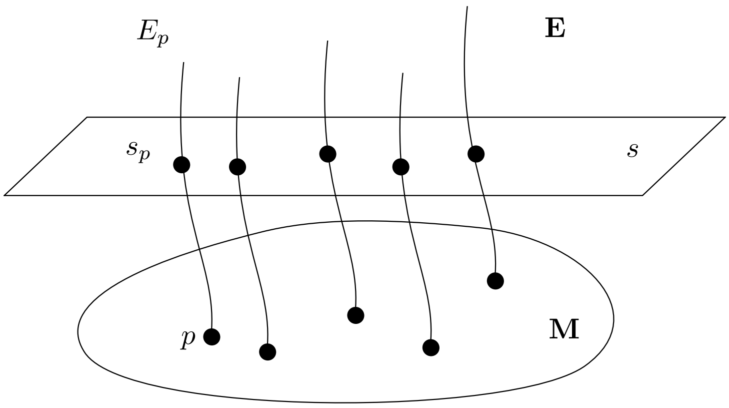


Figure 2.1: Fiber bundle and section

One of the fundamental problem for the theory of vector bundles is the construction of global holomorphic sections of a given bundle, and the main difficulty is there is no *holomorphic partion of unity*, since if a holomorphic function has compact support then naturally it is contant 0.

An important tool is the L^2-method for the $\overline{\partial}-$equation. One can find the basics from Hormander's book. It is interesting that whether or not we can solve the equation depends on the geometry, in particular, the curvature of the bundle.

Definition 2.2.3

Let $\pi^E : E \to M$ and $\pi^F : F \to M$ are holomorphic vector bundles of rank r and s resp. A **bundle map** from E to F is a holomorphic map $f : E \to F$ such that

1. f maps E_p to F_p for any $p \in M$;
2. $f|_{E_p} : E_p \to F_p$ is a linear.

When a bundle map has an inverse bundle map, we will say that two bundles are **isomorphic**.

Another fundamental problem is the *classification problem*. One important tool is the theory of characteristic classes that we shall discuss later.

Also the set of isomorphic classes of holomorphic vector bundles over a given complex manifold has rich structures and is an important invariant for the complex manifold.

Example 2.2.1 (trivial bundle). *For complex manifold M then $M \times \mathbb{C}^r$ with $\pi : M \times \mathbb{C}^r \to M$ is a holomorphic vector bundle over M, called the **product bundle** over M, then the holomorphic bundle that is isomorphic to $X \times \mathbb{C}^r$ is called **trivial bundle**, denoted by $\underline{\mathbb{C}^r}$.*

Example 2.2.2 (holomorphic tangent bundle). *Let M be a complex manidold f dimension n. We shall now construct its "holomorphic tangent bundle" TM as follows :*

Let $p \in M$, we first define the $\mathbb{C}$-module

$$\mathcal{O}_{M,p} := \varinjlim \mathcal{O}_M(U) = \bigsqcup_{U \ni p} \mathcal{O}_M(U) \Big/ \sim,$$

*where the direct limits is taken with respect to open sets with $p \in U$ and the equivalent relation is given by $f \in \mathcal{O}_M(U)$ equivalent to $g \in \mathcal{O}_M(V)$ iff we can find another open set $p \in W \subseteq U \cap V$ such that $f|_W = g|_W$, and $\mathcal{O}_{M,p}$ is called the **stalk** of $\mathcal{O}_M$ at p, an element of $\mathcal{O}_{M,p}$, i.e., a equivalence class $[f]$ is called an **germ of holomorphic function** at p.*

*A **tangent vector** at p is a derivation $X : \mathcal{O}_{M,p} \to \mathbb{C}$, i.e., a $\mathbb{C}$-linear map satisfying the Leibniz rule*

$$\boxed{X_p(fg) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g)}.$$

*The set of tangent vectors at p is easily seen to be a $\mathbb{C}-$vector space. We call it the **holomorphic tangent space** of M at p, denoted by T_pM.*

If $\varphi_i : U_i \to \mathbb{C}^n$ is a holomorphic coordinate chart with $\varphi_i = (z^1, \cdots, z^n)$. Then we can define $\left.\frac{\partial}{\partial z^k}\right|_p \in T_pM$ to be

$$\left.\frac{\partial}{\partial z^k}\right|_p (f) := \frac{\partial (f \circ \varphi_i^{-1})}{\partial z^k}(\varphi_i(p)) \in \mathbb{C},$$

Then one can show that $\boxed{\left\{\left.\frac{\partial}{\partial z^k}\right|_p\right\}_{k=1}^n}$ is a basis of T_pM, using the same way of real manifold.

Let $TM = \bigsqcup_{p \in M} T_pM$, and define $\pi : TM \to M$ in the obvious way. We can make it a holomorphic vector bundle as follows: Let (U_i, φ_i) be a holomorphic chart. Then we can define the local trivialization $\widetilde{\varphi}_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ to be

$$\widetilde{\varphi}_i \left(p, a^k \left.\frac{\partial}{\partial z^k}\right|_p\right) := (p, a^1, \cdots, a^n).$$

This gives a complex structure on TM and at the same times gives a local trivialization of TM over U_i, and if $U_i \cap U_j \neq \varnothing$, then one can easily check if $\varphi_i = (z^1, \cdots, z^n)$ and $\varphi_j = (w^1, \cdots, w^n)$ then

$$\psi_{ij}(p) = (\varphi_i \circ \varphi_j^{-1})_{*,p} = \left(\frac{\partial z^l}{\partial w^k}(\varphi_j(p))\right)_{(l,k)}.$$

*A holomorphic section of TM over U is called a **holomorphic vector field** on U.*

Example 2.2.3 (holomorphic cotangent bundle)**.** *Any $f \in \mathcal{O}_{M,p}$ defines a linear functional on T_pM by $X_p \mapsto X_p(f)$, we call this $\mathrm{d}f|_p \in (T_pM)^* =: T_p^*M$. T_p^*M is called the **holomorphic cotangent space** of M at p. It is easy to see that if (U_i, φ_i) is a holomorphic chart, and $\varphi_i = (z^1, \cdots, z^n)$, then $\{\mathrm{d}z^k|_p\}_{k=1}^n$ is the baisis of T_p^*M dual to $\left\{ \left.\frac{\partial}{\partial z^k}\right|_p \right\}_{k=1}^n$.*

*We can similarly give $T^*M := \bigsqcup_{p\in M} T_p^*M$ a holomorphic bundle structure as TM, called the **holomorphic cotangent bundle** of M.*

*A holomorphic section of T^*M over U is called a **holomorphic 1-form** on U.*

Line Bundles

To have a better understanding of vector bundles, we need to start from some basic vector bundles, trivial bundle is to boring, so the next interesting but not too hard objects are line bundles.

Let $\pi : L \to M$ be a holomorphic line bundle and $\{U_i\}_{i\in I}$ an open cover by trivialization neighborhoods, and $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ the trivialization map. Since $\mathrm{Gl}(1, \mathbb{C}) = \mathbb{C}^*$, now the transition map $\psi_{ij} : U_i \cap U_j \to \mathbb{C}^*$ become non-vanishing holomorphic functions on $U_i \cap U_j$.

Let $s \in \Gamma(M, \mathcal{O}(L))$, then $\varphi_i \circ s|_{U_i} : U_i \to U_i \times \mathbb{C}$ cou;d be represented by a holomorphic function $f_i \in \mathcal{O}(U_i)$, such that $\varphi_i \circ s|_{U_i}(p) = (p, f_i(p))$, note it is $s|_{U_i}$ since s is a global section.

When $U_i \cap U_j \neq \varnothing$, since $s|_{U_i} = s|_{U_j}$ on $U_i \cap U_j$, we have fora any $p \in U_i \cap U_j$:

$$\begin{aligned}(p, f_i(p)) &= \varphi_i \circ s(p) = (\varphi_i \circ \varphi_j^{-1}) \circ \varphi_j(s(p)) \\ &= (\varphi_i \circ \varphi_j^{-1})(p, f_j(p)) = (p, \psi_{ij}(p) f_j(p)).\end{aligned}$$

So we have $f_i = \psi_{ij} f_j$ on $U_i \cap U_j$, i.e., once we have a global holomorphic section s, there exists a family of holomorphic functions $f_i \in \mathcal{O}(U_i)$ satisfying this condition.

On the other hand, once we have such family of holomorphic functions $\{f_i\}_{i\in I}$, one can locally define $s|_{U_i}$, and from $f_i = \psi_{ij} f_j$, we can patch $s|_{U_i}$ together, it is easy to check the global s is actually holomorphic, so the condition and the existence of global holomorphic section is equivalent.

Remark. Using the language of Čech cohomology, we have

$$\exists\, \{f_i\}_{i\in I} \text{ such that } f_i = \psi_{ij} f_j \quad \Longleftrightarrow \quad \{\psi_{ij}\} \text{ is a coboundary},$$

since $\{\psi_{ij}\}$ is already a cocycle, this means that $[\{\psi_{ij}\}] = 0$ in $H^2(M, \mathcal{O})$, so from this we know that the the cohomology describes the obstructions to construct a global holomorphic section.

Example 2.2.4 (Universal line bundle over $\mathbb{CP}^n$)**.** *We define a holomorphic line bundle $\mathbb{U} \to \mathbb{CP}^n$ as follows: As a set,*

$$\mathbb{U} = \{([z], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | v \in [z]\},$$

where we view $[z]$ as the 1-dimensional subspace of $\mathbb{C}^{n+1}$ determined by z. As one can easily check, we can write

$$\mathbb{U} = \{([z], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | v^i z^j - v^j z^i = 0, \forall i, j = 0, \cdots, n\}.$$

From this, it is easy to see that $\mathbb{U}$ is a complex submanifold of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ and hence a complex manifold. The projection onto its first component $\mathbb{CP}^n$ is clearly a holomorphic map, with fiber the 1-dimensional linear subspace of $\mathbb{C}^{n+1}$ generated by $(z^0, \cdots, z^n)$.

Now we check $\mathbb{U}$ is really a line bundle, thus for local triviality, we use the holomorphic charts $\{U_i, \varphi_i\}_{i=0}^n$ defined before, recall $U_i = \{[\boldsymbol{z}] | z^i \neq 0\}$, and

$$\varphi_i([z^0, \cdots, z^n]) = \left(\frac{z^0}{z^i}, \cdots, \widehat{\frac{z^i}{z^i}}, \cdots, \frac{z^n}{z^i} \right),$$

thus on $\pi^{-1}(U_i)$, each $v \in \mathbb{U}_{[\boldsymbol{z}]}$, can be uniquely written as $t \cdot \left(\frac{z^0}{z^i}, \cdots, 1, \cdots, \frac{z^n}{z^i} \right)$, so we define

$$\widetilde{\varphi}_i([\boldsymbol{z}], v) = \widetilde{\varphi}_i(\boldsymbol{z}, t \cdot v_0) = ([\boldsymbol{z}], t) \in U_i \times \mathbb{C}, \quad v_0^i = \left(\frac{z^0}{z^i}, \cdots, 1, \cdots, \frac{z^n}{z^i} \right).$$

So we have $\{\pi^{-1}(U_i), \widetilde{\varphi}_i\}$ is the local triviality of $\mathbb{U}^n$, and note hard to see that the transition function on $U_i \cap U_j$, then we have $\psi_{ij}([\boldsymbol{z}]) = \frac{z^i}{z^j}$, more precisely, suppose $\widetilde{\varphi}_i([\boldsymbol{z}], v) = ([\boldsymbol{z}], t_i)$, and $\widetilde{\varphi_j}([\boldsymbol{z}], v) = ([\boldsymbol{z}], t_j)$, then we know ψ_{ij} from the equation below

$$\psi_{ij} t_j \cdot v_0^i = t_i \cdot v_0^i = v = t_j \cdot v_0^j.$$

Constructing New Bundles From Old Ones

The usual constructions in linear algebra all have counterparts in the category of vector bundles over M. Let E, F be vector bundles over M of rank r and s respectively.

1. **(Direct Sum)**

   The direct sum of E and F is a vector bundle of rank $r + s$ with fiber $E_p \oplus F_p$. Suppose $\{U_i, \varphi_i\}$ and $\{V_j, \psi_j\}$ is local trivialization of E, F respectively, then WLOG there is a refinement of $\mathcal{U}$ and $\mathcal{V}$, and $\boxed{\varphi_i \times \psi_j}$ is the target trivilization.

   Assumen the transition maps are η_{ij} and γ_{ij} respectively, then the transition maps for $E \oplus F$ are precisely $\boxed{\text{diag}\{\eta_{ij}, \gamma_{ij}\}}$.

2. **(Tensor Product)**

   The tensor product of E and F is a vector bundle of rank rs with fiber $E_p \otimes F_p$. Now assume E is a general vector bundle and L is a line bundle, with transition maps ψ_{ij} and η_{ij} respectively, note $\psi_{ij} \in \text{Gl}(r, \mathbb{C})$, and $\eta_{ij} \in \mathbb{C}^*$, thus the transition maps is $\eta_{ij}\psi_{ij}$.

Actually, we have a more general result, suppose ψ_{ij} and η_{ij} are the transition maps of E, F resp., then for any $(p, v \otimes w) \in E \otimes F$, we have

$$v_i \otimes w_i = \Psi_{ij} v_j \otimes w_j, \quad v_i = \psi_{ij} v_j, w_i = \eta_{ij} w_j,$$

so more precisely the transition maps of $E \otimes F$ is $\boxed{\Psi_{ij} = \psi_{ij} \otimes \eta_{ij}}$.

3. **(Hom(E, F))**

   Hom(E, F) is a vector bundle of rank rs with fiber Hom(E_p, F_p). In particular, we define the dual of E to be $E^* :=$ Hom$(E, \underline{\mathbb{C}})$, whose fiber over p is exactly the dual space of E_p, i.e, the $(E_p)^*$. Since there is a natural isomorphism between V and V^*, thus one can easily write the local trivilization of E^*.

   Now we consider the transition functions, suppose ψ_{ij} is the transition maps of E, then from a genreal linear algebra result: If $V \xrightarrow{f} W$ has matrix A, then $W^* \xrightarrow{f^*} V^*$ has matrix A^T. Thus we konw that the transition maps of E^* , $\Psi_{ji} = \psi_{ij}^\mathsf{T}$, so we have $\boxed{\Psi_{ij} = (\psi_{ij}^\mathsf{T})^{-1}}$.

   In general, from linear algebra, we know that Hom$(V, W) \cong V^* \otimes W$, one can check this by writing all the basis down. So we naturally have Hom$(E, F) \cong E^* \otimes F$, thus from this we know that the transition maps are precisely $\boxed{(\psi_{ij}^\mathsf{T})^{-1} \otimes \eta_{ij}}$

Example 2.2.5 (The hyperplane bundle)**.** *Let $\mathbb{U} \to \mathbb{CP}^n$ be the universal line bundle, its dual is usually denoted by $\mathbb{H}$, we call it the* ***hyperplane line bundle****. Another common notation for $\mathbb{H}$ is $\mathcal{O}(1)$, which comes from algebraic geometry. We also write the $\mathbb{H}^k$, or $\mathcal{O}(k)$, short for the k-times tensor product of $\mathbb{H}$, i.e., $\mathbb{H}^k := \mathbb{H} \otimes \cdots \otimes \mathbb{H}$, and $\mathcal{O}(-k) := \mathbb{H}^{-k} = \mathbb{U}^k$.*

Theorem 2.2.1: The section of hyperplane line bundle

Suppose $\mathbb{H}^k \to \mathbb{CP}^n$ be the hyperplane line bundle, and $k > 0$, then

$$\dim_{\mathbb{C}} \Gamma(\mathbb{CP}^n, \mathcal{O}(\mathbb{H}^k)) = \binom{n+k}{k}.$$

Proof. Let $s \in \Gamma(\mathbb{CP}^n, \mathcal{O}(\mathbb{H}^k))$, then from preceeding discusions, we recall two basic facts:

1. For a line bundle L with transition function $\{\psi_{ij}\}$, then s is a global section if an only if there exists $f_i \in \mathcal{O}(U_i)$ such that $\varphi_i \circ s|_{U_i} = (\text{id}, f_i)$, and $f_i = \psi_{ij} f_j$.

2. The transition function of $\mathbb{U}$ is $\psi_{ij} = \dfrac{z^i}{z^j}$, then since $\mathbb{H}$ is the dual of $\mathbb{U}$, then the transition maps are $\dfrac{z^j}{z^i}$, furthermore, for the tensor product , the transition maps Ψ_{ij} of $\mathbb{H}^k$ is actually $\left(\dfrac{z^j}{z^i}\right)^k$.

So from above, we know that s can be represented by $f_i \in \mathcal{O}(U_i)$, where $U_i = \{[\boldsymbol{z}] \in \mathbb{CP}^n | z^i \neq 0\}$. There f_i's satisfy the following equation

$$\boxed{f_i([\boldsymbol{z}]) = \left(\frac{z^j}{z^i}\right)^k f_j([\boldsymbol{z}]), \quad \boldsymbol{z} \in U_i \cap U_j}.$$

Pulling back to $\mathbb{C}^{n+1} - \{0\}$ we can view $(z^i)^k f_i([\boldsymbol{z}])$ as a homogenous function of degree k on $\mathbb{C}^{n+1} - \{z^i = 0\}$, which is also holomorphic. Now the above compatibility condition means that these $(z^i)^k f_i([\boldsymbol{z}])$'s could be glued together to form a holomorphic function F on $\mathbb{C}^{n+1} - \{0\}$, homogenous with degree k.

By *Hartogs extension theorem*, it extends to a holomorphic function $F \in \mathcal{O}(\mathbb{C}^{n+1})$, and from the homogeneity and continuity, we know that $F(0) = 0$. From this we easily conclude that F is a homogeneous polynomial of degree k.

On the other hand, it is easy to see that any homogeneous polynomial of degree k in $\mathbb{C}[z^0, \cdots, z^n]$ determines uniquely a holomorphic section of $\mathbb{H}^k$. So we have

$$\boxed{s \in \Gamma(\mathbb{CP}^n, \mathcal{O}(\mathbb{H}^k)) \overset{1,1}{\longleftrightarrow} \text{homogeneous polynomial of degree } k \in \mathbb{C}[z^0, \cdots, z^n]},$$

since the latter has basis $(z^{i_1})^{k_1} \cdots (z^{i_l})^{k_l}$ with $k_1 + \cdots + k_l = k$, then given $(i_1, \cdots, i_l)$, the numbers of the positive integer solutions is $\binom{k-1}{l-1}$, and the tuple $i_1 < \cdots < i_l$'s number is $\binom{n+1}{l}$, then

$$\dim_{\mathbb{C}} \Gamma(\mathbb{CP}^n, \mathcal{O}(\mathbb{H}^k)) = \sum_{l=1}^{k} \binom{k-1}{l-1} \cdot \binom{n+1}{l} = \binom{n+k}{n},$$

then we finish the proof, hope there is another way to calculate the dimension of the polynomial. ♣

Remark. If $k < 0$, then similary if there is a global section s, then there must have a family of $f_i \in \mathcal{O}(U_i)$ such that $\dfrac{f_i([\boldsymbol{z}])}{(z^i)^k} = \dfrac{f_j([\boldsymbol{z}])}{(z^j)^k}$ on $U_i \cap U_j$, note $\dfrac{f_i([\boldsymbol{z}])}{(z^i)^k}$ is still holomorphic on $\mathbb{C}^{n+1} - \{z^i = 0\}$, thus we can still glue them together and have a holomorphic function F on $\mathbb{C}^{n+1} - \{0\}$, but one can not extend F to 0, which is contradicted to Hartogs extension theorem, then we know that f_i must all vanish, thus when $k < 0$, $\Gamma(\mathbb{CP}^n, \mathcal{O}(\mathbb{H}^k)) = \{0\}$.

Definition 2.2.4: The Picard group

The isomorphic classes of holomorphic line bundles over M is called the **Picard group** of M, and the group operation is given by

$$\boxed{[L_1] \cdot [L_2] := [L_1 \otimes L_2]},$$

and the unit element is $\underline{\mathbb{C}}$, and $[L]^{-1} := [L^*]$, and the group is denoted by $\operatorname{Pic}(M)$.

For $\mathbb{CP}^n$, we have $\operatorname{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$, and any holomorphic line bundle is isomorphic to $\mathcal{O}(k)$ for some $k \in \mathbb{Z}$. However, this is rather deep, we will need sheaf cohomology to prove it.

5. **(Wedge Product)**

   Let E be vector bundles over M of rank r, for $k \in \mathbb{N}$ and $k \leq r$, the degree k wedge product of E is a vector bundle $\wedge^k E$ with fiber $\wedge^k E_p$ at p. The highest degree wedge product $\wedge^r E$ is also called the **determinant line bundle** of E, since its transition functions are precisely $\boxed{\text{der}\psi_{ij}}$.

6. **(Pull back via holomorphic map)**

   Let $E \to M$ be a holomorphic vector bundle of rank r, $f : N \to M$ be a holomorphic map between complex manifolds, then we can define a **pull back** holomorphic vector $f^{\square} E$ over N. In fact, we can simply define the total space of $f^* E$ to be

   $$f^* E := \{(y, (x, v)) \in Y \times E | x = f(y)\},$$

   and $p : f^* E \to Y$ is just the projection to its first component.

   Now we describe $f^* E$ via transition maps: if $\{U_i\}_{i \in I}$ is a trivializing covering of M for E with transition maps $\psi_{ij} : U_i \cap U_j \to \text{GL}(r, \mathbb{C})$, then we choose an open covering $\{V_j\}_{j \in J}$ such that $f(V_j) \subseteq U_i$ for some $i \in I$. We fix a map $\tau : J \to I$ such that $f(V_j) \subseteq U_{\tau(j)}$.Then the transition maps for $f^* E$ with respect to $V_s \cap V_t$, then the transition maps $\boxed{\Psi_{st} = \psi_{\tau(s)\tau(t)} \circ f}$.

2.3 Almost Complex Manifolds

One of our interested problems is :

When can we make a manifold into a complex manifold ?

Recall the reason we want to study manifold is to generalize calculus, so the differention and integration are the most improtant, thus when given a real even oriented manifold, before we give it a complex structure, we need to firstly make its tangent space into a complex vector space , if not, we can not even define what is $\sqrt{-1}$!

Generally, for a $2n$ real vector space, $\sqrt{-1}$ is actually an endomorphism

Definition 2.3.1: complex structure on real vector space

If V is a real vector space of dimension $2n$, then we call a $\mathbb{R}-$linear map $J : V \to V$ such that $J^2 = -\text{id}$ is a **complex structure** on V.

Remark. If V have a complex structure, then V can be regarded as a $\mathbb{C}-$vector space by defining

$$(a + b\sqrt{-1})v := av + bJv, \quad \forall a, b \in \mathbb{R}, v \in V,$$

actually J always exists and not unique, for a basis $e^i = (0, \cdots, 1, \cdots, 0)$, one can check that for arbitary $P \in \text{GL}(2n, \mathbb{R})$, if J can be represented by the matrix

$$P^{-1}\text{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\} P,$$

then J is actually a complex structure on V.

Now for a $2n$ real oriented manifold, for $p \in M$, we can define a real tangent vector at p and the corresponding real tangent space at p, $T_p^{\mathbb{R}}M$. In terms of coordinate chart$\varphi = (x^1, y^1, \cdots, x^n, y^n)$, we have $T_p^{\mathbb{R}}M = \text{span}_{\mathbb{R}}\{\partial_{x^i}|_p, \partial_{y^i}|_p\}_{i=1}^n$.We can give $\bigsqcup T_p^{\mathbb{R}}M$ a structure of $\mathbb{R}$-vector bundle of rank $2n$, called the **real tangent bundle** of M, and denoted by $T^{\mathbb{R}}M$.

Now we hope we can make the tangent space into a complex vector space, thus

Definition 2.3.2: Almost complex manifold

Let M be a real orientable differential manifold of dimension $2n$. An **almost complex structure** on M is a bundle map $J : T^{\mathbb{R}}M \to T^{\mathbb{R}}M$ satisfying $J^2 = -\text{id}$. And a real manifold with such almost complex structure is called **almost complex manifold**.

Remark. If one ignore the "bundle map" condition, one may think each manifold is almost complex manifold, since we can always define $J_p^2 = -\text{id}$ at $T_p^{\mathbb{R}}M$. But the truth is that J the bundle map actually is a $(1,1)$ tensor field, thus it depends smoothly on $p \in M$.

Globally, having an almost complex structure means that one can define the J_p in any patch and glue them together without encountering obstructions or singularities. There are examples where such obstructions appear; the most notable is the four-sphere S^4. It is known not to allow for an almost complex structure (see e.g. Steenrod, 1951), hence S^4 is not an almost complex manifold.

Theorem 2.3.1

Complex manifolds are almost complex.

Proof. Naturally, for a complex manifold, locally we have a coordinate chart $(z^1, \cdots, z^n)$, and holomorphic tangent bundle T^hM, with basis $\left\{ \frac{\partial}{\partial z^k}\Big|_p \right\}_{k=1}^n$, then suppose $z^k = x^k + \sqrt{-1}y^k$, then we can view M as a real $2n$ oriented manifold with local coordinate $(x^1, y^1, \cdots, x^n, y^n)$, then we define

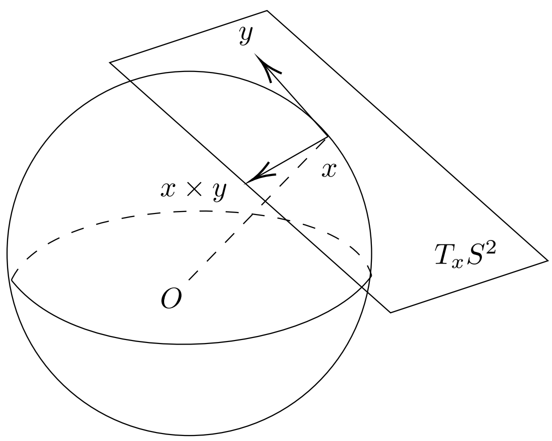
$$J : T^{\mathbb{R}}M \to T^{\mathbb{R}}M, \quad J\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}, \quad J\frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k},$$

one can easily check J is really a bundle map, thus is a almost complex structure. ♣

Definition 2.3.3

If an almost complex structure is induced from a complex structure , we will call it **integrable**.

Example 2.3.1. *For S^2, we can define $J : T^{\mathbb{R}}S^2 \to T^{\mathbb{R}}S^2$ as follows: we identify $T_x^{\mathbb{R}}S^2$ with the subspace of $\mathbb{R}^3$:$T_x^{\mathbb{R}}S^2 \cong \{y \in \mathbb{R}^3 | x \cdot y = 0\}$. Then we can define $J_x : T_x^{\mathbb{R}}S^2 \to T_x^{\mathbb{R}}S^2$ by $J_x(y) := x \times y$. On can check that this is an integrable almost complex structure, induced by the complex structure of $S^2 \cong \mathbb{CP}^1$, and J actually means the rotation of 90° clockwise.*



Example 2.3.2. *For S^6, we have a similar almost complex structure given by* ***wedge product*** *in $\mathbb{R}^7$. Note that the wedge product in $\mathbb{R}^3$ can be defined as the product of purely imaginary quaternions. To define this wedge product in $\mathbb{R}^7$, we shall use Cayley's theory of octonions.*

We write $\mathbb{H} \cong \mathbb{R}^4$ the space of ***quaternions*** *$q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ with $a, b, c, d \in \mathbb{R}$, satsifying*

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$$

and

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

Then this multiplication is still associative but not commutative. For $q \in \mathbb{H}$, we define $\bar{q} := a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$, then $|q|^2 = q\bar{q}$.

Now we define the space of ***octonians****, $\mathbb{O} \cong \mathbb{R}^8$, as $\mathbb{O} := \{x = (q_1, q_2) | q_1, q_2 \in \mathbb{H}\}$. The multiplication is defined by*

$$(q_1, q_2)(q_1', q_2') := (q_1 q_1' - \bar{q_2'} q_2, q_2' q_1 + q_2 \bar{q_1'}).$$

We also define $\bar{x} := (\bar{q_1}, -q_2)$, then we still have $x\bar{x} = x \cdot x = |x|^2$, here the $\cdot$ means the ususal inner product in $\mathbb{R}^8$. Note this multiplication is even not associative.

We identify $\mathbb{R}^7$ as the space of purely imaginary octonians. If $x, x' \in \mathbb{R}^7$, we define $x \times x'$ as the imaginary part of xx'. Then we can check that $xx = -|x|^2$, $x \times x' = -x' \times x$, and $(x \times x') \cdot x'' = x \cdot (x' \times x'')$.

From this, one can define an almost complex structure on $S^6 \subseteq \mathbb{R}^7$ in a similary way as S^2: identify $T_x S^6$ with $\{y \in \mathbb{R}^7 | x \cdot y = 0\}$, then define

$$J_x(y) := x \times y.$$

One can prove that this almost complex structure is not integrable. (Ref: Calabi: Construction and properties of some 6-dimensional almost complex manifolds)

Remark. For spheres of even dimension $2n$, it is known (Borel-Serre) that there are no almost complex structures unless $n = 1, 3$. A modern proof of this fact using characteristic classes can be found in P. May's book on algebraic topology. It is generally believed that there are no integrable almost complex structures on S^6, however S.T. Yau has a different conjecture saying that one can make S^6 into a complex manifold. This is still open.

Now we return back to the discussion about *complexified the tagent space*, it is easiliy to be found that the way we complexify is not quite well ,because we use J as $\sqrt{-1}$, but we hope the coordinate is z, and there is no z in our previous discussion.

So now to make the coordinate from $(x^1, y^2, \cdots, x^n, y^n)$ to $(z^1, \cdots, z^n)$, note that if the manifold is holomorphic, then there is no need to consider $\overline{z}$, but now we only have smoothness, so we just simply complexify V to get $V_{\mathbb{C}} := V \otimes \mathbb{C}$. We also extend J $\mathbb{C}-$linearly to $V_{\mathbb{C}}$, again $J^2 = -\text{id}$.

There is a direct sum decomposition of $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, which are eigenspaces of J resp. In fact we have a very precise description of $V^{1,0}$ and $V^{0,1}$:

$$\boxed{V^{1,0} = \{v - \sqrt{-1}Jv | v \in V\}, \quad V^{0,1} = \{v + \sqrt{-1}Jv | v \in V\}}.$$

Now apply this to $(T^{\mathbb{R}}M, J)$ for a manifold with an almost complex structure: define the complexified tangent bunlde to be $T^{\mathbb{C}}M := T^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ and we have the decomoposition $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, which are the $\sqrt{-1}$ and $\sqrt{-1}$ eigenspaces of J, respectively.

When J is integrable, $T^{1,0}M$ is locally generated by $\left\{ \dfrac{\partial}{\partial z^k} \right\} \Big|_{k=1}^{n}$, so we can identify it with T^hM, the holomorphic tangent bundle of complex manifold M.

2.4 de Rham Cohomology and Dolbeault Cohomology

From now on , we will always assume M is a complex manifold, and now we consider the holomorphic cotangent space $\bigwedge^{p,q} T^*M$, which is locally generated by

$$\mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\overline{z^{j_1}} \wedge \cdots \wedge \mathrm{d}\overline{z^{j_q}},$$

we denote the smooth sections of $\bigwedge^{p,q} T^*M$ over an open set U is denoted by $\mathcal{A}^{p,q}(U)$, and

$$\mathcal{A}^k(U) = \Gamma\left(U, \bigwedge^k T^*M\right) = \Gamma\left(U, \bigoplus_{p+q=k} \bigwedge^{p,q} T^*M\right).$$

Definition 2.4.1

Naturally, we can define the exterior differential operator $\mathrm{d} : \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$, and furthermore, we define the operators

$$\partial := \Pi^{p+1,q} \circ \mathrm{d} : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p+1,q}(U)$$

and

$$\overline{\partial} := \Pi^{p,q+1} \circ \mathrm{d} : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p,q+1}(U),$$

where $\Pi^{p,q}$ is the projection maps from $\mathcal{A}^{p+q}(U)$ to $\mathcal{A}^{p,q}(U)$.

Remark. When the beginner firstly meet these three operators may be quite confused, *why we define such* ∂? Formally, it is not wrong to view z^k and $\overline{z^k}$ just $2n$ different variables, then ∂ and $\overline{\partial}$ seem just to distinguish those two kinds of variable, why? This is because the definition of holomorphic, we always hope one thing f is holomorphic ,then we need $\overline{\partial} f = 0$, i.e, to avoid $\mathcal{A}^{0,q}$ things occur.

Now a smooth section of $\bigwedge^{p,q} T^*M$ over a coordinate open set U is of the forms

$$\eta = \sum_{1 \le i_1 \le \cdots \le i_p \le n, 1 \le j_1 \le \cdots \le j_q \le n} a_{i_1 \cdots i_p, \bar{j}_1 \cdots \bar{j}_q} \mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\overline{z^{j_1}} \wedge \cdots \wedge \mathrm{d}\overline{z^{j_q}},$$

where $a_{i_1 \cdots i_p, \bar{j}_1 \cdots \bar{j}_q} \in \boxed{C^\infty(U; \mathbb{C})}$. we write $\eta = \sum_{|I|=p, |J|=q} a_{I\bar{J}} \mathrm{d}z^I \wedge \mathrm{d}z^{\bar{J}} \in \mathcal{A}^{p,q}(U)$ for short.

In this case, we have

$$\begin{aligned} \mathrm{d}\eta &= \sum_{I,J} \mathrm{d}a_{I\bar{J}} \wedge \mathrm{d}z^I \wedge \mathrm{d}z^{\bar{J}} \\ &= \sum_{I,J} \partial a_{I\bar{J}} \wedge \mathrm{d}z^I \wedge \mathrm{d}z^{\bar{J}} + \sum_{I,J} \overline{\partial}_{I\bar{J}} \wedge \mathrm{d}z^I \wedge \mathrm{d}z^{\bar{J}} \\ &\in \quad \mathcal{A}^{p+1,q}(U) \quad \oplus \quad \mathcal{A}^{p,q+1}(U). \end{aligned}$$

So we always have $\boxed{\mathrm{d} = \partial + \overline{\partial}}$.

Since we always have $\mathrm{d}^2 = 0$, we have $0 = \partial^2 + \overline{\partial}^2 + (\partial\overline{\partial} + \overline{\partial}\partial)$, acting on $\mathcal{A}^{p,q}(M)$. Comparing types, we get

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

We can define from these identities several differential cochain complexes:

1. **(de Rham complex)**

$$0 \to \mathcal{A}^0(M) \xrightarrow{\mathrm{d}} \mathcal{A}^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{A}^{2n}(M) \to 0.$$

From this we define the **de Rham cohomology** (with coefficient $\mathbb{C}$)

$$\boxed{H^k_{dR}(M, \mathbb{C}) := \text{Ker } \mathrm{d}|_{\mathcal{A}^k(M)} \Big/ \mathrm{d}\mathcal{A}^{k-1}(M)}.$$

Its dimension b_k is called the ***k*-th Betti number** of M.

2. **(Dolbeault complex)**

$$0 \to \mathcal{A}^{p,0}(M) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{p,n}(M) \to 0.$$

From this we define the **Dolbeault cohomology**

$$\boxed{H^{p,q}_{\overline{\partial}}(M) := \text{Ker } \overline{\partial}|_{\mathcal{A}^{p,q}(M)} \Big/ \overline{\partial}\mathcal{A}^{p,q-1}(M)}.$$

Its dimension $h^{p,q}$ is called the **Hodge number** of M, they are important invariants of the complex manifold.

3. **(Holomorphic de Rham complex)**

$$0 \to \Omega^0(M) \xrightarrow{\mathrm{d}=\partial} \Omega^1(M) \xrightarrow{\mathrm{d}=\partial} \cdots \xrightarrow{\mathrm{d}=\partial} \Omega^n(M) \to 0,$$

where $\Omega^*(M)$ locally generated by $\mathcal{O}(M) \otimes \{\mathrm{d}z^k\}_{k=1}^n$, since always we hvae $\overline{\partial} : \mathcal{O}(M) \mapsto 0$, then $\mathrm{d} = \partial + \overline{\partial} = \partial$, we define the holomorphic **de Rham cohomology**

$$\boxed{H^k_{dR}(X, \Omega) := \text{Ker}\left(\Omega^k(M) \xrightarrow{\mathrm{d}} \Omega^{k+1}(M)\right) \Big/ \mathrm{d}\Omega^{k-1}(M)}.$$

The relation between these cohomology theories, as well as computational tools will be discussed when we finish sheaf cohomology theory and Hodge theorem.

3 Sheaf Theory

3.1 Presheaves and Sheaves

Definition 3.1.1: Presheaf

A **presheaf** $\mathcal{F}$ of abealian groups (or vector spaces, rings, etc.) over a topological space M consists of an abealian group (or vector spaces, rings, etc.) $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ for every open subset $U \subset M$ and a group homomorphism (resp. linear map, ring homomorphism,etc.) for each pair $V \subset U$, $r_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$, called **restriction homomorphism**, satisfying

1. $r_U^U = \mathrm{id}_{\mathcal{F}(U)} : \mathcal{F}(U) \to \mathcal{F}(U)$;
2. for any $W \subset V \subset U$, we have $r_W^U = r_V^U \circ r_W^V : \mathcal{F}(U) \to \mathcal{F}(W)$.

Remark. One who is familiar to the Category theory will quickly realize, the presheaf is actually a contravariant functor from (M, ι) to Abel (or Vect, Ring,resp.), where the category of (M, ι) has the objects : all of the open subsets of M, and the morphisms: the inclusion map of $\iota : V \subset U$.

Definition 3.1.2: Section, Stalk

An element of $\mathcal{F}(U)$ is uaually called a **section** of $\mathcal{F}$ over U. We also defined the **stalk** of $\mathcal{F}$ at a point $p \in M$ to be

$$\mathcal{F}_p := \varinjlim \mathcal{F}(U) = \bigsqcup_{U \ni p} \mathcal{F}(U) \Big/ \sim$$

where the direct limit is taken with respect to open sets $p \in U$, and $s \in \mathcal{F}(U)$ is equivalent to $t \in \mathcal{F}(V)$ iff we can find another open set $p \in W \subset U \cap V$ such that $r_W^U(s) = r_W^V(t)$. The image of $s \in \mathcal{F}(U)$ in $\mathcal{F}_p$ is an equivalence class, denoted by s_p, called the **germ** of s.

Example 3.1.1 (continuous function presheaf)**.** *For $\mathcal{F} = \mathcal{C}_M^0$, the presheaf of continuous function on M. More precisely, $\mathcal{C}_M^0(U)$ is the ring of all continuous maps $f : U \to \mathbb{R}$, and the restriction homomorphisms are really the restriction.*

*And generally, one may call the stalk of $\mathcal{C}_M^0$ at a point p is the **function germ**, one should note that they denote the same thing. And if $s_p = t_p$, then it does not only mean $s(p) = t(p)$, instead, it is a much sronger condition, means that we can find a neighborhood V of p such that $s|_V = t|_V$.*

Now we continue the discussion of the presheaf $\mathcal{C}_M^0$, the following two additional conditions are naturally satisfied: We let $U = \bigcup U_i$ be a union of open subsets $U_i \subset M$, then

1. If $f, g \in \mathcal{C}^0_M(U)$ with $\boxed{r^U_{U_i}(f) = r^U_{U_i}(g)}$ for all i, then $f = g$;
2. If functions $f_i \in \mathcal{C}^0_M(U_i)$ are given for all i such that $\boxed{r^{U_i}_{U_i \cap U_j}(f_i) = r^{U_j}_{U_i \cap U_j}(f_j)}$ for any j, then we can patch them together, i.e, there exists a function $f \in \mathcal{C}^0_M(U)$ with $r^U_{U_i}(f) = f_i$ for all i.

Those propoerties are very essential, but not all presheaf can satisfy the conditions above, so now we hope to study the better presheaf, we will use the propeties above as two more axioms:

Definition 3.1.3: Sheaf

A presheaf of abelian groups $\mathcal{F}$ over M is called a **sheaf**, if it satisfies the following two prperties:

(S1) Assume we have a family of open sets $U_i \subset U$, $i \in I$ and $\bigcup_i U_i = U$. If $s, t \in \mathcal{F}(U)$ satisfies $\boxed{r^U_{U_i}(s) = r^U_{U_i}(t)}$, for all $i \in I$, then $\boxed{s = t}$;

(S2) Assume we have a family of open sets $U_i \subset U$, $i \in I$ and $\bigcup_i U_i = U$. If we also have a family of sections $s_i \in \mathcal{F}(U_i)$, for all $i \in I$, satisfying $\boxed{r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)}$ whenever $U_i \cap U_j \neq \varnothing$, then there is a section $s \in \mathcal{F}(U)$ such that $r^U_{U_i}(s) = s_i$, for all $i \in I$.

Remark. Note that by (S1), the section in (S2) is also unique.

Example 3.1.2 (preshaef but not sheaf)**.** *Let G be a given abelian group, we define the constant presheaf over M to be $G_{\text{pre}}(U) := G$ for any non-empty open set $U \subset M$, and $r^U_V = \text{id}$ for any non-empty pair $V \subset U$. But generally, it is not a presheaf, for example, suppose $\boxed{M = U \sqcup V}$, then for $s \in G_{\text{pre}}(U)$, then it is actually a element in group G, and $t \in G_{\text{pre}}(V)$, with $s \neq t$, then from (S2), there exists $g \in G$ such that $s = g = t$, a contradiction.*

There are more examples of sheaves on the complex manifold M:

1. $\mathcal{O}_X$ is the shaef of commutative rings of holomorphic functions over M, and $\mathcal{O}_X(U)$ denotes the holomorphic functions on U, we also call it the **structure sheaf** of M.
2. $\mathcal{E}$ is the sheaf of commutative rings of smooth functions over M, and $\mathcal{E}(U) = \mathcal{C}^\infty(U, \mathbb{C})$.
3. If $\pi : E \to M$ is a holomorphic vector bundle, and $\mathcal{O}(E)$ is the sheaf of all sections, more precisely, $\mathcal{O}(E)(U)$ denotes the all holomorhic sections over U, it is actually a $\mathcal{O}_X-$module.

Definition 3.1.4: Sheaf Homomorphism

Let $\mathcal{F}$ and $\mathcal{G}$ be two (pre)sheaves. A **(pre)sheaf homomorphism** $\varphi : \mathcal{F} \to \mathcal{G}$ is given by group homomorphisms for each U open, $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, such that whenever $V \subset U$, we have

$$\boxed{(r^{\mathcal{G}})^U_V \circ \varphi_U = \varphi_V \circ (r^{\mathcal{F}})^U_V}.$$

Remark. More precisely, we have the a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ {\scriptstyle (r^{\mathcal{F}})^U_V}\downarrow & & \downarrow{\scriptstyle (r^{\mathcal{G}})^U_V} \\ \mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V) \end{array}$$

Once a homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ of (pre)sheaves of abealian groups is given, one constructs the associated presheaves Kerφ, Imφ and Coker(φ) which are defined in the obivious way, for example

$$\text{Coker}(\varphi)(U) := \text{Coker}(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)) = \mathcal{G}(U)/\text{Im}\varphi_U.$$

But one should note that they are all presheaves, but only Ker(φ) in general is a sheaf, but Im(φ) and Coker(φ) aren't.

Proof. *Why* Ker(φ) *is a sheaf?* Now we check by definition:

(S1) Assume $U = \bigcup_i U_i$, and we have a section $s \in \text{Ker}(\varphi)(U) := \mathcal{K}(U)$, and for each $i \in I$, we have

$$(r^{\mathcal{K}})^U_{U_i}(s) = 0,$$

since we have a natural presheaf homomorphism, induced from the inclusion $\iota : \mathcal{K} \to \mathcal{F}$,

$$\begin{array}{ccc} \mathcal{K}(U) & \xrightarrow{\iota_U} & \mathcal{F}(U) \\ {\scriptstyle (r^{\mathcal{K}})^U_{U_i}}\downarrow & & \downarrow{\scriptstyle (r^{\mathcal{F}})^U_{U_i}} \\ \mathcal{K}(U_i) & \xrightarrow[\iota_{U_i}]{} & \mathcal{F}(U_i) \end{array}$$

and then we note that $s \in \mathcal{K}(U)$, naturally $\boxed{\iota_U(s) = s \in \mathcal{F}(U)}$, so we have

$$(r^{\mathcal{F}})^U_{U_i}(s) = (r^{\mathcal{F}})^U_{U_i} \circ (\iota_U)(s) = \iota_{U_i} \circ (r^{\mathcal{K}})^U_{U_i}(s) = 0,$$

then from $\mathcal{F}$ is a sheaf, then we know that $s = 0$, then $\mathcal{K}$ satisfies the sheaf axiom I.

(S2) Now we assume $s_i \in \mathcal{K}(U_i) = \text{Ker}(\varphi_{U_i})$, and $(r^{\mathcal{K}})^{U_i}_{U_i \cap U_j}(s_i) = (r^{\mathcal{K}})^{U_j}_{U_i \cap U_j}(s_j)$, so we have

$$\begin{aligned} & (r^{\mathcal{K}})^{U_i}_{U_i \cap U_j} \circ \iota_{U_i}(s_i) = (r^{\mathcal{K}})^{U_j}_{U_i \cap U_j} \circ \iota_{U_j}(s_j) \\ \Rightarrow\ & \iota_{U_i \cap U_j} \circ (r^{\mathcal{F}})^{U_i}_{U_i \cap U_j}(s_i) = \iota_{U_i \cap U_j} \circ (r^{\mathcal{F}})^{U_j}_{U_i \cap U_j}(s_j) \\ \Rightarrow\ & \boxed{(r^{\mathcal{F}})^{U_i}_{U_i \cap U_j}(s_i) = (r^{\mathcal{F}})^{U_j}_{U_i \cap U_j}(s_j)}, \end{aligned}$$

the last equation comes from the injectivity of $\iota_{U_i \cap U_j}$, then since $\mathcal{F}$ is a sheaf, then there exists a section $s \in \mathcal{F}(U)$, such that $(r^{\mathcal{F}})^U_{U_i}(s) = s_i$, naturally one can show that $\varphi_U(s) = 0$ then $s \in \mathcal{K}(U)$ simiarly, we omit the last proof since it is boring and trivial.

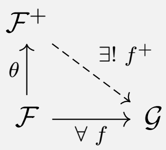
Finally we show $\mathrm{Ker}(\varphi)$ is really a sheaf. ♣

Remark. One may feel afraid of sheaf since it may be too abstract for the beginner, but one should note that, whenever we use sheaf acting on an open set U, things turn to be quite clear and easy, because now they are just group theory.

Now since there are a lot of presheaves which are not sheaves, so to get more sheaves, a natural idea is to *sheafificate* the presheaves:

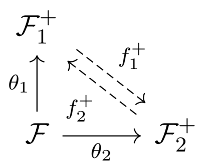
Theorem 3.1.1: Sheafification

For any presheaf $\mathcal{F}$ over M, there is a unique (up to isomorphism) sheaf $\mathcal{F}^+$ and a homomorphism $\theta : \mathcal{F} \to \mathcal{F}^+$ satisfying the following **universal property**:



i.e., for any sheaf $\mathcal{G}$ over M and any homomorphism of presheaves $f : \mathcal{F} \to \mathcal{G}$, there is a unique homomorphism of sheaves $f^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $f = f^+ \circ \theta$. If $\mathcal{F}$ is already a sheaf, then θ is an isomorphism. $\mathcal{F}^+$ is called the **shefification** of $\mathcal{F}$.

Remark. By universal property, if $\mathcal{F}_1^+$ with θ_1 and $\mathcal{F}_2^+$ with θ_2 are both the sheafifications of $\mathcal{F}$,



then we know from the diagram that the induced f_1^+ and f_2^+ give the isomorphism, so the shefification for any presheaf is unique .

Proof. The most direct proof is to define $\mathcal{F}^+(U)$ explicitely: a map

$$\widetilde{s} : U \to \bigsqcup_{p \in U} \mathcal{F}_p$$

is an element of $\mathcal{F}^+(U)$ if and only if:

1. $\pi \circ \widetilde{s} = \mathrm{id}_U$, i.e. $\widetilde{s}(p) \in \mathcal{F}_p$ for all $p \in U$, here $\pi : \mathcal{F}_p \mapsto p$;

2. For any $p \in U$, there is an open neighborhood $p \in V \subset U$ and a $s \in \mathcal{F}(V)$ such that for any $q \in V$, $\widetilde{s}(q)$ equals s_q, the stalk of s at q.

Now we check that $\mathcal{F}^+$ is really the sheafification of $\mathcal{F}$:

(Step1) we will consturct a homomorphism $\theta : \mathcal{F} \to \mathcal{F}^+$, recall when we define a homomorphism, we actually define a group homomorphism for any $\mathcal{F}(U)$, so it is suffices to define θ_U, thus for any $s \in \mathcal{F}(U)$, then $\theta_U(s) = \widetilde{s}$, where for each $p \in U$, $\boxed{\widetilde{s}(p) = s_p}$.

Then we check that θ is really a homomorphism, i.e, for any $V \subset U$, to check $\theta_V \circ (r^{\mathcal{F}})^U_V = (r^{\mathcal{F}^+})^U_V \circ \theta_U$, then for s and $\forall q \in V$, they both equal to s_q, so we know that it is a morphism.

(Step2) we will show that $\mathcal{F}^+$ is really a sheaf, using the morphism θ on can easily prove (S1) and (S2) these two axioms, it is so similar to the proof of Ker(φ), so we omit here.

(Step3) we now show the universal property, similarly, we need to construct group homomorphism f^+_U for each U open. Naturally, we define $\boxed{f^+_U(\widetilde{s}) = f(s) \in \mathcal{G}(U)}$, one should check the existence of s, then the uniqueness is trivial to check.

Now we finish the proof, but one can see that I left so much things to be checked, because when I was typing the proof, I gradually felt boring about it. ♣

Remark. One can define shefification from **étalé space**:

Definition 3.1.5

From $\mathcal{F}$, we define a topological space, called the **étalé space** associated to $\mathcal{F}$:

$$\widetilde{\mathcal{F}} := \bigsqcup_{p \in M} \mathcal{F}_p.$$

We have a natural map $\pi : \widetilde{\mathcal{F}} \to M$. The topology on $\widetilde{\mathcal{F}}$ is given as follows: for any open subset $U \subset M$ and an element $s \in \mathcal{F}(U)$, let

$$[U, s] := \{s_p \mid p \in U\} \subset \widetilde{\mathcal{F}}$$

and they forms a baisis $\mathcal{B}$ of the topology.

Remark. One can refer GTM 81 for complete proof. Then for any open $U \subset M$, define $\mathcal{F}^+(U) := \{s : U \to \widetilde{\mathcal{F}} \text{ is continuous } | \pi \circ s = \mathrm{id}_U\}$.

Remark. If one still can not see sheafication, you should just remeber, *a sheaf is completely determined by its stalks*, so the key point is $\boxed{\mathcal{F}^+_p = \mathcal{F}_p}$, we only need to care the local information.

Example 3.1.3. *For the constant presheaf G_{pre} over a manifold M, denotes its sheafification $\underline{G}$. Then the elements of $\underline{G}(U)$ consists of locally constant maps from U to the abealian group, and it is denoted by **constant sheaf**. If you are confused with this concept, you can refer handwiki.*

Example 3.1.4. *Let M be a complex manifold, we define a presheaf $\mathcal{M}_{\text{pre}}$ over M as follows: for open set $U \subset M$, elements of $\mathcal{M}_{\text{pre}}(U)$ are quotients of holomorphic functions on U, with denominator not* *identically zero on any connected component of U. Its sheafification $\mathcal{M}$ is the sheaf of* ***meromorphic functions****. Elements of $\mathcal{M}(U)$ are called meromorphic functions on U.*

Example 3.1.5 (Skyscraper Sheaf)**.** *Suppose $\mathcal{S}$ is a presheaf of a fixed* Top *T_1 space X, and fixed a point $p \in X$, then define $\mathcal{S}(U) = G$, if $p \in U$, otherwise $\mathcal{S}(V) = \{0\}$, and actually it is a sheaf, and*

$$\mathcal{S}_p = G, \quad \mathcal{S}_q = \{0\}, \quad \forall q \neq p.$$

Proof. We check the sheaf axioms, suppose $\mathcal{U} = \cup U_i$, for (S1), suppose $s \in \mathcal{S}(\mathcal{U})$, and $s|_{U_i} = 0$, if $p \notin \mathcal{U}$, then trivially $s = 0$, otherwise if $p \in \mathcal{U}$ and $s \in G^*$, then we know that $s|_{U_i} \neq 0$, a contradiction.

Now we check (S2), if we have $\{s_i\}$ and $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, then if $\mathcal{U} = \cup U_i \cup U_j$, and $p \in U_i$, $p \notin U_j$ for all i, j, thus if there exists $U_{ij} \neq \varnothing$, then s_i all equals to 0, so $s = 0$ is as desired, and the another case is really trivial, so we omit. ♣

Example 3.1.6. *Suppose $U \subset \mathbb{C}$, then if $f \in \mathcal{O}(U)$, then f_p is the Taylor expansion of f at p, and $\mathcal{O}_p$ is isomorphic to the convergent power seires at p.*

Corollary 3.1.1

For bounded holomorphic function presheaf $\mathcal{B}$, we have $\mathcal{B}^+ = \mathcal{O}$.

Proof. The key point is $\mathcal{B}_p^+ = \mathcal{O}_p$, they are all isomorphic to the convergent power seires at p. ♣

Definition 3.1.6: Soft Sheaf

A sheaf $\mathcal{F}$ over X is called **soft**, if for any closed subset $K \subseteq X$, the resrtiction map

$$\mathcal{F}(X) \to \mathcal{F}(K) := \varinjlim \mathcal{F}(U)$$

where U takes the all open sets contain K, is surjective.

3.2 Sheaf Cohomology

In this section, we always assume X is a manifold and $\mathcal{F}$ is a sheaf of Abel.

Motivation:the Mittag-Leffler Problem

Sheaf is a useful tool to describe the obsructions to solve global problems when we can always solve a local one. (Recall the discussion we made in finding a global holomorphic section for line bundle)

To illustrate this point more precisely, we come back to the Mittag-Leffler problem on a Riemann surface M (Recall it is a complex manifold of dimension 1):

Problem 1. Suppose we are given finitely many points $p_1, \cdots, p_m \in M$, and for each p_i we are given a Laurent polynomial $\sum_{k=1}^{n_i} c_k^{(i)} z^{-k}$. We can view this an element of $\mathcal{M}_p/\mathcal{O}_p$. We want to find a global mermorphic function on M whose

1. poles are precisely those p_i's
2. with the given Laurent polynomial as its principal part of p_i.

This problem is always solvable locally : we can find a locally finite open covering $\mathcal{U} = \{U_i | i \in I\}$ of M such that each U_i contains at most one of the p_i's, and $f_i \in \mathcal{M}(U_i)$ such that the only poles of f_i are those of $\{p_i\}$ contained in U_i with the principal part equals the given Laurent polynomial.

The problem is that we can not patch them together : if $U_i \cap U_j \neq \varnothing$, there is no reason to have $f_i = f_j$. So we have to define

$$\boxed{f_{ij} := f_j - f_i} \in \mathcal{O}(U_i \cap U_j) =: \mathcal{O}(U_{ij})$$

and view the totality of these f_{ij}'s as the obstruction to solve the problem.

Now by our choiceof f_i, $f_{ij} \in \mathcal{O}(U_{ij})$ is because there are no poles for f_i and f_j on $U_i \cap U_j$, and note that we have

$$f_{ij} + f_{ji} = 0 \quad \text{on } U_i \cap U_j$$
$$f_{ij} + f_{jk} + f_{ki} = 0 \quad \text{on } U_i \cap U_j \cap U_k$$

and we call this the **cocycle** condition and $\{f_{ij}\}$ is a **Čech cocycle** for the sheaf $\mathcal{O}$ w.r.t. the cover $\mathcal{U}$.

Now when can we solve the Mittag-Leffler problem on M? We can solve it if we can modify the f_i by a holomorphic function $h_i \in \mathcal{O}(U_i)$ such that $\widetilde{f_i} := f_i - h_i$ will patch together . (This is because we only wish the principal part is as our desired, we do not really care about the holomorphic part)

This means that $\widetilde{f}_i = \widetilde{f}_j$ on $U_i \cap U_j$, equivalently,

$$\boxed{f_{ij} = h_j - h_i},$$

naturally, $\{h_i - h_j\}$ is also a cocycle for the sheaf $\mathcal{O}$, but since h_i's are all holomorphic, we call them a **Čech coboundary**. So we get the conclusion that we can solve the Mittag-Leffler problem if

$$\boxed{\textit{the Čech cocycle } \{f_{ij}\} \textit{ is a coboundary.}}$$

Čech Cohomology

The discussion above motivates the introduction of the following Čech cohomology of a sheaf $\mathcal{F}$ with respect to a locally finite cover $\mathcal{U}$ of X. We first define the chain groups:

Definition 3.2.1

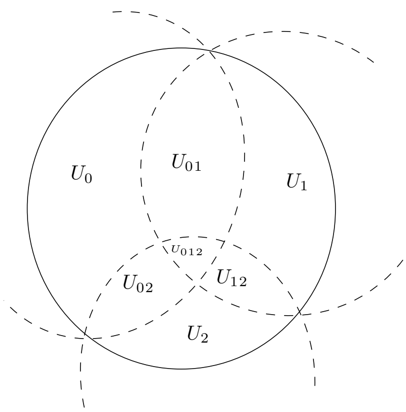
Given sheaf $\mathcal{F}$ and locally finite cover $\mathcal{U}$, we define

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{F}) &:= \prod_{i \in I} \mathcal{F}(U_i) \\ C^1(\mathcal{U}, \mathcal{F}) &\subset \prod_{(i_0, i_1) \in I^2} \mathcal{F}(U_{i_0} \cap U_{j_0}) \\ &\cdots \\ C^p(\mathcal{U}, \mathcal{F}) &\subset \prod_{(i_0, \cdots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \\ &\cdots \end{aligned}$$

where $\{\sigma_{i_0 \cdots i_p}\}$ is in $C^p(\mathcal{U}, \mathcal{F})$ if and only if

1. whenever $i_k = i_l$ for some $k \neq l$, we have $\sigma_{i_0 \cdots i_p} = 0$;
2. For any permutation $\tau \in S_{p+1}$, we have $\sigma_{i_{\tau(0)} \cdots i_{\tau(p)}} = (-1)^{\text{sgn}\tau} \sigma_{i_0 \cdots i_p}$.

Remark. Note that we always define $\mathcal{F}(\varnothing) = \{0\}$, and write $U_{i_0 \cdots i_p}$ short for $U_{i_0} \cap \cdots \cap U_{i_p}$, one should always note that in our definition, the $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ will have $|I|!/(|I| - p - 1)!$ components, but in most books, they only consider $i_0 < \cdots < i_p$ and thus have only $\binom{|I|}{p+1}$ components.



Example 3.2.1. *If $X = D^1 = \{z||z| \leq 1\}$ and is covered by $\mathcal{U}$ as right, so we have $\sigma \in C^0(\mathcal{U}, \mathcal{F})$, then* $\boxed{\sigma = (\sigma_0, \sigma_1, \sigma_2)}$, *$\eta \in C^1(\mathcal{U}, \mathcal{F})$, then* $\boxed{\eta = (\eta_{01}, \eta_{02}, \eta_{12})}$, *$\gamma \in C^2(\mathcal{U}, \mathcal{F})$, then $\gamma = (\gamma_{012})$.*

Definition 3.2.2: Coboundary Maps

We define the coboundary operator $\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ to be :

$$(\delta\sigma)_{i_0\cdots i_{p+1}} := \sum_{j=0}^{p+1}(-1)^j \cdot (r^{\mathcal{F}})^{U_{i_0\cdots\widehat{i_j}\cdots i_{p+1}}}_{U_{i_0\cdots i_{p+1}}}\left(\sigma_{i_0\cdots\widehat{i_j}\cdots i_{p+1}}\right).$$

Remark. One should not be afraid of this formula, since it is natural if you view $r^{\mathcal{F}}$ just as the restriction, more precisely, one can compare this formula with the one in singular homology.

Proposition 3.2.1

We have $\delta \circ \delta = 0$, so we have a cochain complex $\{C^*(\mathcal{U}, \mathcal{F}), \delta\}$:

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots.$$

The proof is direct, we omit here and we can define the space of **Čech $p-$cocycles**

$$Z^p(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}\delta \subset C^p(\mathcal{U}, \mathcal{F}),$$

and the the space of **Čech $p-$coboundaries**

$$B^p(\mathcal{U}, \mathcal{F}) = \delta C^{P-1}(\mathcal{U}, \mathcal{F}) \subset C^p(\mathcal{U}, \mathcal{F}),$$

and the **Čech cohomology** with respect to $\mathcal{U}$

$$\boxed{H^p(\mathcal{U}, \mathcal{F}) := Z^p(\mathcal{U}, \mathcal{F})/B^p(\mathcal{U}, \mathcal{F})}.$$

Now we study H^0 and H^1 more precisely :

- If $[f] \in H^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F})$, then $f = (f_i)_{i\in I} \in Z^0(\mathcal{U}, \mathcal{F})$ is a cocycle, i.e., $\delta f = 0$, since

$$(\delta f)_{ij} = r^{U_j}_{U_{ij}}(f_j) - r^{U_i}_{U_{ij}}(f_i) = 0$$

  which means that $r^{U_j}_{U_{ij}}(f_j) = r^{U_i}_{U_{ij}}(f_i)$, since $f_i \in \mathcal{F}(U_i)$, thus from (S2), we get a global section $\widetilde{f}$ and from (S1) the uniqueness, we know $f = \widetilde{f}$ is a global section ,thus $f \in \mathcal{F}(X)$. So $H^0(\mathcal{U}, \mathcal{F})$ is in fact independent of $\mathcal{U}$ and we have a canonical isomorphism

$$\boxed{H^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)}.$$

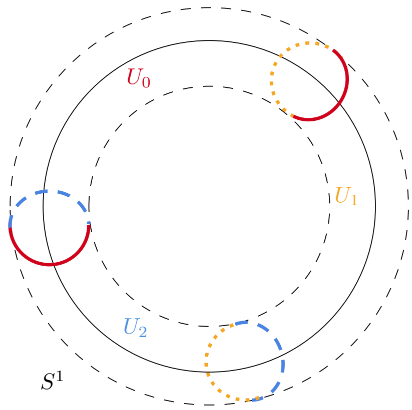
- If $[g] \in H^1(\mathcal{U}, \mathcal{F})$, then $g = (g_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$ is a cocycle, so from $\delta g = 0$, we actually have

$$f_{ij} + f_{ji} = 0 \quad \text{on } U_i \cap U_j$$
$$f_{ij} + f_{jk} + f_{ki} = 0 \quad \text{on } U_i \cap U_j \cap U_k$$

  This is precisely the "cocycle condition" we met before. However, this time the cohomology may depend on the cover

Now we calculate a more precise example, and it comes from Bott *Exercise 10.7*:

Example 3.2.2 (Cohomology with Twisted Coefficients). *Let $\mathcal{F}$ be the presheaf on S^1 which associates to every open set the group $\mathbb{Z}$. We define the restriction homomorphism on the cover $\mathcal{U} = \{U_0, U_1, U_2\}$ as the figure right by*



$$r_{01}^0 = r_{01}^1 = 1,$$
$$r_{12}^1 = r_{12}^2 = 1,$$
$$r_{02}^2 = -1, r_{02}^0 = 1,$$

where r_{ij}^i denotes the restriction from U_i to $U_i \cap U_j$. Now we calculate $H^*(\mathcal{U}, \mathcal{F})$, recall we have

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} 0 \to \cdots.$$

- *For $H^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F})$, suppose $f = (f_0, f_1, f_2) \in C^0(\mathcal{U}, \mathcal{F}) = \mathbb{Z}^3$,now we find δf:*

$$(\delta f)_{01} = r_{01}^1(f_1) - r_{01}^0(f_0) = f_1 - f_0$$
$$(\delta f)_{02} = r_{02}^2(f_2) - r_{02}^0(f_0) = -f_2 - f_0$$
$$(\delta f)_{12} = r_{12}^2(f_2) - r_{12}^1(f_1) = f_2 - f_1,$$

  so if $\delta f = 0$, then we have $f_1 = f_0$, $f_2 + f_0 = 0$ and $f_2 = f_1$, which means that $f_0 = f_1 = f_2 = 0$, then we have $f = 0$, thus $\mathrm{Ker}\delta = 0$, then $\boxed{H^0(\mathcal{U}, \mathcal{F}) = 0}$.

- *For $H^1(\mathcal{U}, \mathcal{F}) = C^1(\mathcal{U}, \mathcal{F})/\mathrm{Im}\delta$, and in $\mathrm{Im}\delta$, suppose $h_{01} = f_1 - f_0$, $h_{02} = -f_2 - f_0$, $h_{12} = f_2 - f_1$, since $h_{01} + h_{02} + h_{12} = -2f_0 \in 2\mathbb{Z}$, thus we know that $\boxed{H^1(\mathcal{U}, \mathcal{F}) = \mathbb{Z}_2}$.*

Remark. One should note that the reason why $H^0 \neq \mathcal{F}(S^1)$ is because $\mathcal{F}$ is not a sheaf, here we define the twisted restriction homomorphism is only for exercise!

Now we consider if there are two different covers :

Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a locally finite refinement of $\mathcal{U}$. This means we have a map $\tau : I \to J$ (not unique) such that $V_j \subset U_{\tau(j)}$, then we have a homomorphism $\Phi_{\mathcal{V}}^{\mathcal{U}} : H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{U}, \mathcal{F})$ induced by

$$C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F}), \quad (\sigma_{i_0 \cdots i_p}) \mapsto \left(\sigma_{\tau(j_0)\cdots\tau(j_p)}|_{V_{j_0 \cdots j_p}}\right).$$

One can prove that $\Phi_{\mathcal{V}}^{\mathcal{U}}$ is in fact independent of the choice of the map τ.

The cohomology of X with coefficients sheaf $\mathcal{F}$ is defined to be the direct limit:

$$H^p(X, \mathcal{F}) := \varinjlim H^p(\mathcal{U}, \mathcal{F}) = \bigsqcup_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F}) \Big/ \sim$$

where two cohomology classes $[(\sigma_{i_0\cdots i_p})] \in H^p(\mathcal{U},\mathcal{F})$ and $[(\eta_{j_0\cdots j_p})] \in H^p(\mathcal{V},\mathcal{F})$ are equivalent if we can find a common refinement $\mathcal{W}$ of $\mathcal{U},\mathcal{V}$ such that

$$\boxed{\Phi^{\mathcal{U}}_{\mathcal{W}}\left([(\sigma_{i_0\cdots i_p})]\right) = \Phi^{\mathcal{V}}_{\mathcal{W}}\left([(\eta_{j_0\cdots j_p})]\right)}.$$

Thus an element of $H^p(\mathcal{U},\mathcal{F})$ is an equivalent class of Čech cohomology classes, represented by an element of $H^p(\mathcal{U},\mathcal{F})$, for some cover $\mathcal{U}$. But in many cases, in particular all the sheaves we use in this notes, there exists sufficiently fine cover $\mathcal{U}$ such that $H^p(\mathcal{U},\mathcal{F}) \cong H^p(X,\mathcal{F})$. (If one is familiar with Bott Tu, then will soon realize it is similar to the good cover for smooth manifold)

Now before we end this section, we will give a more detailed discussion of H^1 :

Proposition 3.2.2

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $\Phi^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F})$ is injective, and hence so is the induced homomorphism $H^1(\mathcal{U},\mathcal{F}) \to H^1(X,\mathcal{F})$, i.e., we can simply write

$$\boxed{H^1(X,\mathcal{F}) = \bigcup H^1(\mathcal{U},\mathcal{F})}.$$

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$, $\mathcal{V} = \{V_\alpha\}_{\alpha\in\Gamma}$ and $\tau : \Gamma \to I$ be a map such that $V_\alpha \subset U_{\tau(\alpha)}$, suppose we have $[(f_{ij})] \in H^1(\mathcal{U},\mathcal{F})$ satisfies $\Phi^{\mathcal{U}}_{\mathcal{V}}\left([(f_{ij})]\right) = 0$. Then consider a common refinement of $\mathcal{U}$ and $\mathcal{V}$

$$\mathcal{W} := \{W_{i\alpha} := U_i \cap V_\alpha \neq \varnothing | i \in I, \alpha \in \Gamma\},$$

then we have $\Phi^{\mathcal{U}}_{\mathcal{W}}([f]) = \Phi^{\mathcal{V}}_{\mathcal{W}} \circ \Phi^{\mathcal{U}}_{\mathcal{V}}([f]) = 0$, this implies that (f_{ij}) is a cocycle, and

$$\Phi^{\mathcal{U}}_{\mathcal{W}}([f]) = [((f_{ij}|_{W_{i\alpha}\cap W_{i\beta}}))] = 0 \quad \Rightarrow \quad \boxed{(f_{ij}|_{W_{i\alpha}\cap W_{i\beta}})}$$

is a coboundary. So we can find $h_{i\alpha} \in \mathcal{F}(W_{i\alpha})$ such that on $W_{i\alpha} \cap W_{j\beta}$, we have

$$f_{ij}|_{W_{i\alpha}\cap W_{i\beta}} = h_{j\beta} - h_{i\alpha}.$$

Since naturally $f_{ii} = 0$ by definition, we must have $0 = h_{i\alpha}|_{W_{i\alpha}\cap W_{i\beta}} - h_{i\beta}|_{W_{i\alpha}\cap W_{i\beta}}$. Since $\{W_{i\alpha}\}_{\alpha\in\Gamma}$ is an open covering of U_i, by (S2), we can find a $h_i \in \mathcal{F}(U_i)$ such that $h_i|_{W_{i\alpha}} = h_{i\alpha}$.

Now consider the covering of $U_i \cap U_j$ by $\boxed{U_i \cap U_j \cap V_\alpha = W_{i\alpha} \cap W_{j\alpha}}$, since

$$\begin{aligned} f_{ij}|_{W_{i\alpha}\cap W_{i\beta}} &= h_j|_{W_{i\alpha}\cap W_{i\beta}} - h_i|_{W_{i\alpha}\cap W_{i\beta}} \\ &= (h_j|_{U_i\cap U_j} - h_i|_{U_i\cap U_j})|_{W_{i\alpha}\cap W_{i\beta}} \\ &= (\delta(h_i))_{ij}|_{W_{i\alpha}\cap W_{i\beta}}, \end{aligned}$$

then from (S1) the uniqueness, we have $(f_{ij}) = \delta(h_i)$, equivalently, $[(f_{ij})] = 0$, i.e, $\phi^{\mathcal{U}}_{\mathcal{V}}$ is injective. ♣

Now we finish our discussion about Picard group in §2.2

Theorem 3.2.1

Let X be a complex manifold, then we have $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of nowhere vanishing holomorphic functions.

Remark. Recall $\mathrm{Pic}(X)$ is the holomorphic line bundle over X under isomorphism claasification.

Proof. Intuitively, the isomorphism comes from cocycle condition , now we offer more details

1. Given a holomorphic line bundle L with local trivializing $\mathcal{U}$, we get a cocycle $\{\psi_{ij}\}$, and ψ_{ij} : $U_i \cap U_j \to \mathrm{GL}(1, \mathbb{C})$ holomorphically, then we know that $(\psi_{ij}) \in \mathcal{O}^*(U_{ij})$, furthermore, $[(\psi_{ij})] \in H^1(\mathcal{U}, \mathcal{O}^*) \subset H^1(X, \mathcal{O}^*)$, from this we know $\mathrm{Pic}(X) \subseteq H^1(X, \mathcal{O}^*)$.

2. On the other hand, if L is isomorphic to L', we can assume that they have common trivializing coverings $\mathcal{U}$, with cocycles $\{\psi_{ij}\}$ and $\{\psi'_{ij}\}$ resp. The bundle isomorphism map gives $\lambda_i \in \mathcal{O}^*(U_i)$, such that $\psi'_{ij}\lambda_j = \lambda_i\psi_{ij}$ (check !) , this implies that $[(\psi_{ij})] = [(\psi'_{ij})]$, from this we know that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*)$.

One will also find that it is easily to verify this is actually a group isomorphism. ♣

3.3 Fundamental Results for Sheaf Cohomology

Recall that a morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves over X induces for each point $p \in X$ a homomorphism of stalks: $f_p : \mathcal{F}_p \to \mathcal{G}_p$. We call a sequence of morphisms of sheaves an **exact sequence** if the induced sequence on stalks is so for each pint p.

Theorem 3.3.1: Short exact sequence induces long exact sequence

If we have a short exact sequence for sheaves of abelian groups over X

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0,$$

then we have a long exact sequence for cohomologies

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(X,\mathcal{F}) & \longrightarrow & H^0(X,\mathcal{G}) & \longrightarrow & H^0(X,\mathcal{H}) \\
 & \swarrow & & & & & \\
 & & H^1(X,\mathcal{F}) & \longrightarrow & H^1(X,\mathcal{G}) & \longrightarrow & H^1(X,\mathcal{H}) \\
 & & & & \cdots & & \\
 & & H^p(X,\mathcal{F}) & \longrightarrow & H^p(X,\mathcal{G}) & \longrightarrow & H^p(X,\mathcal{H}) \\
 & & \ldots \swarrow & & & &
\end{array}$$

Before we prove this theorem, we need to firstly have a better understanding of the short exact sequence, more precisely, we need know more about sheaf injective and surjective morpism!

Definition 3.3.1

Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf homomorphism, then we call it is

- injective, if the sheaf $\mathrm{Ker}\varphi = 0$;
- surjective, if the sheafication $\mathrm{Im}\varphi = \mathcal{G}$.

But the sheaf is not always convenient to use, then when we follow the main idea of sheaf: *a sheaf is completely determined by its stalks*, we have the following

Proposition 3.3.1: or definition

TFAE (the following are equivalent):

- $\varphi : \mathcal{F} \to \mathcal{G}$ is injective;
- $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective;

- $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective.

Proof. The main reason has contained in the proof of the Kerφ is sheaf. ♣

Proposition 3.3.2

TFAE (the following are equivalent):

- $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective;
- For any $\tau \in \mathcal{G}(U)$, there exists an open cover $\mathcal{U} = \cup U_i$ of U and $s_i \in \mathcal{F}(U_i)$ such that $\tau|_{U_i} = \varphi_{U_i}(s_i)$;
- $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective.

Remark. Generally, we can not have φ_U is surjective, an example is $\mathcal{O} \overset{\exp}{\to} \mathcal{O}^*$, the surjectivity strongly depends on the topological information of U, i.e., the obstructions.

Remark. So in short, when we have a short exact sequence of sheaf $0 \to \mathcal{F} \overset{f}{\to} \mathcal{G} \overset{g}{\to} \mathcal{H} \to 0$, then equivalently , we have

$$\boxed{0 \to \mathcal{F}_p \overset{f}{\to} \mathcal{G}_p \overset{g}{\to} \mathcal{H}_P \to 0}, \quad \forall p \in X,$$

and we have $0 \to \mathcal{F}(U) \overset{f_U}{\to} \mathcal{G}(U) \overset{g_U}{\to} \mathcal{H}(U)$ is exact, $U \subseteq X$ and is open.

The proof of theorem 3.3.1 is quite long and boring, and it is not a good way for calculation, we will use the corollary below more frequently:

Theorem 3.3.2: Abstract de Rham theorem

Suppose we have an exact sequence of the form :

$$0 \to \mathcal{F} \to \mathcal{F}^0 \overset{\partial^0}{\to} \mathcal{F}^1 \overset{\partial^1}{\to} \cdots \to \mathcal{F}^i \overset{\partial^i}{\to} \cdots$$

where each sheaf $\mathcal{F}^i$ satisfies $\boxed{H^p(X, \mathcal{F}^i) = 0}$, for all $p \geq 1$, this is called an **acyclic resolution of $\mathcal{F}$**, then $H^*(X, \mathcal{F})$ is isomorphic to the cohomology of the cochain complex

$$0 \to \mathcal{F}^0(X) \overset{\partial^0_X}{\to} \mathcal{F}^1(X) \overset{\partial^1_X}{\to} \cdots \to \mathcal{F}^i(X) \overset{\partial^i_X}{\to} \cdots,$$

i.e, we actually have the isomorphism

$$H^p(X, \mathcal{F}) \cong \frac{\operatorname{Ker}\left(\mathcal{F}^p(X) \overset{\partial^p_X}{\to} \mathcal{F}^{p+1}(X)\right)}{\operatorname{Im}\left(\mathcal{F}^{p-1}(X) \overset{\partial^{p-1}_X}{\to} \mathcal{F}^p(X)\right)}.$$

4

Differential Geometry of Vector Bundles

4.1 Metrics, Connections and Curvatures

Definition 4.1.1: Hermitian Metrics

Let $E \to X$ be a complex ($\mathbb{C}^\infty$) vector bundle of rank r over a smooth manifold X. A **smooth Hermitian metric** on E is an assignment of Hermitian inner products

$$h_p(\cdot, \cdot) = \langle \cdot, \cdot \rangle_p$$

on each fiber E_p, such that for any smooth sections ξ, η over U, then $h(\xi, \eta) \in C^\infty(U, \mathbb{C})$.

Remark. Recall Hermitian inner product h means that $h(au, bv) = a \cdot \bar{b} \cdot h(u, v)$, i.e., it is $\mathbb{C}-$linear for the first component, and conjugate $\mathbb{C}-$linear for the second.

Let U is a local trivialization neighborhood of E via $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$, then we can define r smooth sections of E over U:

$$e_i(p) := \varphi_U^{-1}(p, 0, \cdots, 0, 1, 0, \cdots, 0), \quad \forall 1 \leq i \leq r.$$

Then at any point p of U, $\{e_i(p)\}_{i=1}^r$ is a basis of E_p. We call $\{e_i\}$ a local frame of E over U. Note that E is a holomorphic bundle and (U, φ_U) a holomorphic trivialization, then these $e_i's$ are also holomorphic sections, and we call it a **holomorphic frame**.

Using local frame, we have local representation of a metric, if ξ is a smooth section over U, then we can write in a unique way $\xi = \xi^i e_i$, with $\xi^i \in C^\infty(U; \mathbb{C})$, the smooth complex valued function. Now we define the smooth fuctions $h_{i\bar{j}} := h(e_i, e_j)$, then we have

$$\boxed{h(\xi, \eta) = h(\xi^i e_i, \eta^j e_j) = h_{i\bar{j}} \xi^i \eta^j}.$$

Now compare to the Riemannian geometry, we need to define connections, since for real version, $\nabla_X Y$ is still a tangent vector, so actually, we can view ∇Y as a TM valued $1-$form.

So now suppose $\pi : E \to M$ be a complex vector bundle on M, we denote by $\mathcal{A}^i(E)$ the sheaf of $i-$forms with values in E, i.e.,

$$\mathcal{A}^i(E)(U) := \mathcal{A}^i(U) \otimes E,$$

and recall $\mathrm{A}^0 = \mathcal{C}^\infty$, the smooth complex function, so for real manifold,

$$\nabla : \mathcal{A}^0(TM) \to \mathcal{A}^1(TM), \quad Y \in TM \mapsto \nabla Y = \omega^i \otimes \partial_i,$$

where ω^i and ∂_i satisfies $\omega^i(X)\partial_i = \nabla_X Y$.

Now we generalize this definition to vector bundle, the motivation is we want to differentiate sections of E (view sections as functions, you will see this is really natural), but it cannot be realized cononically, so we still need differential forms, a natural way is consider their tensors .

Definition 4.1.2: Connection

A **connection** on a smooth rank r complex vector bundle over a manifold X is a map $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which satisfies

1. ∇ is $\mathbb{C}-$linear, i.e, $\nabla(a\xi + b\eta) = a\nabla\xi + b\nabla\eta$, for any $\xi, \eta \in \mathcal{A}^0(E)(U)$;
2. (Leibniz rule) $\nabla(f \otimes \xi) = \mathrm{d}f \otimes \xi + f\nabla\xi$, for any $f \in \mathcal{A}^0(U)$, $\xi \in \mathcal{A}^0(E)(U)$.

Remark. One should note that $\mathcal{A}^i(E)$ is a sheaf, so when we consider sections, they actuall live in $\mathcal{A}^i(E)(U)$, but we usually omit them, one should be clear.

Now we have a local representation of a conncetion : If $\{e_i\}$ is a local frame, then we define a family of local smooth $1-$forms $\boxed{\theta_i^j \in \mathcal{A}^1(U)}$ satisfying :

$$\nabla e_i = \theta_i^j \otimes e_j.$$

Sometimes we just write $\nabla e_i = \theta_i^j e_j$ for short and omit the tensor operator, we call these $\{\theta_i^j\}$ **connection 1-forms**. More generally, for $\xi = \xi^i e_i = \xi^i \otimes e_i$, we have

$$\begin{aligned}\nabla(\xi^i e_i) &= \mathrm{d}\xi^i \otimes e_i + \xi^i \nabla e_i \\ &= \mathrm{d}\xi^i \otimes e_i + \xi^i \theta_i^j e_j \\ &= (\mathrm{d}\xi^i + \xi^j \theta_j^i) \otimes e_i.\end{aligned}$$

Remark. Regard ξ^i as a column vector, and for θ_j^i as a matrix $\theta = (\theta_j^i)_{(i,j)}$, so formally ,we have

$$\nabla\xi = \nabla \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^r \end{pmatrix} = \begin{pmatrix} \mathrm{d}\xi^1 \\ \vdots \\ \mathrm{d}\xi^r \end{pmatrix} + \begin{pmatrix} \theta_1^1 & \cdots & \theta_r^1 \\ \vdots & & \vdots \\ \theta_1^r & \cdots & \theta_r^r \end{pmatrix} \cdot \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^r \end{pmatrix} = (\mathrm{d} + \theta) \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^r \end{pmatrix},$$

so if we identify ξ with the column vector ξ^i, then we can write $\boxed{\nabla = \mathrm{d} + \theta}$ (Physicists' notation).

Now we can extend the action of ∇ to bundle valued differential forms, i.e., we define $\nabla : \mathcal{A}^i(E) \to \mathcal{A}^{i+1}(E)$, more precisely we have

$$\boxed{\nabla(\omega \otimes \xi) := \mathrm{d}\omega \otimes \xi + (-1)^i \omega \wedge \nabla\xi},$$

for any $\omega \in \mathcal{A}^k(U)$, $\xi \in \mathcal{A}^0(E)(U)$, and $\omega \otimes \xi \in \mathcal{A}^k(E)(U)$, one should note that $\omega \wedge \nabla\xi$ actually denotes that ω wedges the 1-form component of $\nabla\xi$.

Remark. one can see that ∇ is really the generalization of d the exterior differential.

Definition 4.1.3: Curvature

We define the curvature of ∇ to be $\Theta := \nabla^2 : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$.

The most improtant property of curvature is the linearity of smooth functions, if $f \in C^\infty$, and $\xi \in \mathcal{A}^0(E)$, then we actually have

$$\begin{aligned}\Theta(f\xi) &= \nabla(\mathrm{d}f\xi + f\nabla\xi)\\ &= \mathrm{d}(\mathrm{d}f)\xi + (-1)\cdot \mathrm{d}f \wedge \nabla\xi + \mathrm{d}f \wedge \nabla\xi + f\nabla^2\xi\\ &= f\Theta(\xi).\end{aligned}$$

Locally if we define the 2-forms $\Theta_i^j \in \mathcal{A}^2(U)$ by

$$\Theta(e_i) = \Theta_i^j \otimes e_j.$$

Then we have

$$\Theta(\xi) = \Theta(\xi^i e_i) = \xi^i \Theta(e_i) = \xi^j \Theta_j^i e_i.$$

Now we consider the local representation of the curvature, i.e., we represent Θ_j^i in terms of θ_j^i:

$$\begin{aligned}\Theta_j^i e_i = \nabla^2(e_j) &= \nabla(\theta_j^l e_l)\\ &= \mathrm{d}\theta_j^l e_l - \theta_j^l \wedge \nabla e_l\\ &= \mathrm{d}\theta_j^i e_i - \theta_j^l \wedge \theta_l^i e_i\\ &= (\mathrm{d}\theta_j^i + \theta_l^i \wedge \theta_j^l)e_i.\end{aligned}$$

so we actually have

$$\Theta_j^i = \mathrm{d}\theta_j^i + \theta_l^i \wedge \theta_j^l.$$

or $\boxed{\Theta = \mathrm{d}\theta + \theta \wedge \theta}$ for short, here we view d acts on a martix as acts on each component.

Proposition 4.1.1: Curvature forms VS Curvature tensor

For any $X, Y \in TM$, we have $\boxed{\Theta(\xi)(X,Y) = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi}$.

Proof. It is known that the both sides are funtional linear for X, Y, ξ, so we assume $X = \partial_i$ and $Y = \partial_j$, $\xi = e_\alpha$, so we have $\Theta(\xi) = \Theta_\alpha^\beta e_\beta$, so $\Theta(\xi)(X,Y) = \Theta_\alpha^\beta(\partial_i, \partial_j)e_\beta$.

Note that $\nabla_Y(\xi) = \nabla\xi(Y) = \theta^\beta_\alpha(Y)e_\beta$, so

$$\begin{aligned}\nabla(\nabla_Y(\xi)) &= \nabla(\theta^\beta_\alpha(Y)e_\beta)\\ &= \mathrm{d}\theta^\beta_\alpha(Y)e_\beta + \theta^\beta_\alpha(Y)\nabla e_\beta,\end{aligned}$$

then we know that

$$\begin{aligned}\nabla_X\nabla_Y\xi &= (\mathrm{d}\theta^\beta_\alpha(Y))(X)e_\beta + \theta^\beta_\alpha(Y)\theta^\gamma_\beta(X)e_\gamma\\ &= \left((\mathrm{d}\theta^\gamma_\alpha(Y))(X) + \theta^\beta_\alpha(Y)\theta^\gamma_\beta(X)\right)e_\gamma.\end{aligned}$$

Thus we have

$$\begin{aligned}&\left(\nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi\right)^\gamma\\ =\ &\mathrm{d}\theta^\gamma_\alpha(Y))(X) + \theta^\beta_\alpha(Y)\theta^\gamma_\beta(X)\\ &- \mathrm{d}\theta^\gamma_\alpha(X))(Y) - \theta^\beta_\alpha(X)\theta^\gamma_\beta(Y)\\ &- \theta^\gamma_\alpha([X,Y])\\ =\ &X(\theta^\gamma_\alpha(Y)) - Y(\theta^\gamma_\alpha(X)) - \theta^\gamma_\alpha([X,Y])\\ &+ \theta^\gamma_\beta(X)\theta^\beta_\alpha(Y) - \theta^\gamma_\beta(Y)\theta^\beta_\alpha(X)\\ =\ &\left(\mathrm{d}\theta^\gamma_\alpha + \theta^\gamma_\beta\wedge\theta^\beta_\alpha\right)(X,Y),\end{aligned}$$

so from $\Theta^\gamma_\alpha = \mathrm{d}\theta^\gamma_\alpha + \theta^\gamma_\beta\wedge\theta^\beta_\alpha$, we finish the proof. ♣

Remark. So one can see that the curvature form is really the curvature tensor in Riemanian geometry.

Now we changes the frame and give the transition representation : Suppose $\{f_i\}$ is another local frame on U, then we can write

$$f_i = a^j_i e_j,$$

where (a^j_i) is a $\mathrm{GL}(r,\mathbb{C})-$valued smooth function on U. (When both frames are local holomorphic frames of a holomorphic bundle, then (a^j_i) is a $\mathrm{GL}(r,\mathbb{C})-$valued holomorphic function on U.)

The new conncetion forms and curvature forms are denoted by $\widetilde{\theta}$ and $\widetilde{\Theta}$, we have

$$\begin{aligned}\widetilde{\theta}^j_i f_j = \nabla f_i &= \nabla(a^k_i e_k)\\ &= \mathrm{d}a^k_i e_k + a^k_i\nabla e_k\\ &= \mathrm{d}a^k_i e_k + a^k_i\theta^j_k e_j\\ &= \left(\mathrm{d}a^k_i + \theta^k_j a^j_i\right)e_k,\end{aligned}$$

and since the left hand side equals $\widetilde{\theta}^j_i a^k_j e_k$, so we have

$$a^k_j\widetilde{\theta}^j_i = \mathrm{d}a^k_i + \theta^k_j a^j_i,$$

and in matrix forms we have $a\widetilde{\theta} = \mathrm{d}a + \theta a$, i.e., we can write as following for short

$$\boxed{\widetilde{\theta} = a^{-1}\mathrm{d}a + a^{-1}\theta a}.$$

Now we consider changing frames of curvature : from the above formula, we get

$$\begin{aligned}
\widetilde{\Theta} &= \mathrm{d}\widetilde{\theta} + \widetilde{\theta} \wedge \widetilde{\theta} \\
&= \mathrm{d}\left(a^{-1}\mathrm{d}a + a^{-1}\theta a\right) + \left(a^{-1}\mathrm{d}a + a^{-1}\theta a\right) \wedge \left(a^{-1}\mathrm{d}a + a^{-1}\theta a\right) \\
&= \mathrm{d}a^{-1} \wedge \mathrm{d}a + \mathrm{d}a^{-1} \wedge \theta a + a^{-1}\mathrm{d}\theta a - a^{-1}\theta \wedge \mathrm{d}a \\
&\quad + \left(a^{-1}\mathrm{d}a + a^{-1}\theta a\right) \wedge \left(a^{-1}\mathrm{d}a + a^{-1}\theta a\right) \\
&= -(a^{-1}\mathrm{d}a \cdot a^{-1}) \wedge \mathrm{d}a - (a^{-1}\mathrm{d}a \cdot a^{-1}) \wedge \theta a + a^{-1}\mathrm{d}\theta a - a^{-1}\theta \wedge \mathrm{d}a \\
&\quad + (a^{-1}\mathrm{d}a \cdot a^{-1}) \wedge \mathrm{d}a + (a^{-1}\mathrm{d}a \cdot a^{-1}) \wedge \theta a + a^{-1}\theta \wedge \mathrm{d}a + a^{-1}\theta \wedge \theta a \\
&= a^{-1}(\mathrm{d}\theta + \theta \wedge \theta)a = \boxed{a^{-1}\Theta a},
\end{aligned}$$

where the main trick is the transition formula

$$\boxed{\mathrm{d}a^{-1} = -a^{-1}\mathrm{d}a \cdot a^{-1}},$$

where it comes from $0 = \mathrm{d}I_r = \mathrm{d}(a \cdot a^{-1}) = \mathrm{d}a \cdot a^{-1} + a\mathrm{d}a^{-1}$.

Remark. However, if we use local representation, we can have a shorter proof with the linearity of curvature form:

$$\begin{aligned}
\widetilde{\Theta}_i^j f_j &= \widetilde{\Theta}(f_i) = \Theta(a_i^k e_k) \\
&= a_i^k \Theta(e_k) = a_i^k \Theta_k^j e_j,
\end{aligned}$$

i.e., we have $\widetilde{\Theta}_i^j a_j^k = a_i^j \Theta_j^k$, note when we write it in martix component, we have $a_j^k \widetilde{\Theta}_i^j = \Theta_j^k a_i^j$, then

$$\boxed{\widetilde{\Theta} = a^{-1}\Theta a}.$$

Above the discussion, we know that Θ is invariant under similar transformation, so from *Morita*, we know that, we want to study topological invariant of vector bundles , since curvature form, a $r \times r$ matrix, with each component is a 2-form, although it may not be globally defined, but we can patch local curvature together, if we find something invariant under similar transfomation of matrix , for example, determinat and trace. So we have

Definition 4.1.4: Chern Form

We can construct a family of globally defined differential forms:

$$c(E, \nabla) := \det\left(I_r + \frac{\sqrt{-1}}{2\pi}\Theta\right) := 1 + c_1(E, \nabla) + \cdots + c_r(E, \nabla),$$

where $c_k(E, \nabla) \in \mathcal{A}^{2k}(X)$ is called the **$k-$th Chern form** of E associated to ∇.

Remark. We will use Chern form to define Chern class later.

In physicists' language, a connection is a **field**, the curvature is the **strength** of the field, and choosing a local frame is called **fixing the gauge**. The reason for these names comes from H. Weyl's work, rewriting Maxwell's equations. The **vector potential** and **scalar potential** together form the connection 1-form, and the curvature 2-form has 6 components, consisting the components of the electric field and the magnetic field, and in short

$$\boxed{\text{Yang-Milles Theory}}$$
$$||$$
$$\boxed{\text{Differential Geometry of Vector Bundles}}$$

4.2 Chern Connection on Holomorphic Vector Bundles

Since there are a lot of connections, we need to find the most speical and interesting one, like Levi-Civta connection in Riemmanian geometry, so firstly, we consider holomorphic vector bundles , and then we need the connection satisfying more necessary conditions:

Theorem 4.2.1: Chern Connection

On a given holomorphic vector bundle E with a smooth Hermitian metric h, there is a unique connection ∇, called **Chern connection** satisfying the following two additional conditions:

1. (*Compatibility with the metric*) If ξ, η are two smooth sections, then we have
$$\mathrm{d}h(\xi,\eta) = h(\nabla\xi,\eta) + h(\xi,\nabla\eta). \tag{4.1}$$
2. (*Compatibility with the complex structure*) If ξ is a holomorphic section of E, then $\nabla\xi$ is a $E-$valued (1,0)-form.

Remark. One may feel confused when see (4.1), but the fact is that h acts on E, and $\nabla\xi$ is a tensor with E and differential forms, so we generalize h so thta $h(\nabla\xi,\eta)$ means that h acts on $E-$valued part and η, the differential part of $h(\nabla\xi,\eta)$ is the differential part of $\nabla\xi$.

Proof. The proof naturally contains two parts, uniqueness and existence:

(Part1) Let $\{e_i\}_{i=1}^r$ be a local holomorphic frame, and the connection $1-$form with respect to this frame is $(\theta^i_j)_{1\le i,j\le r}$, satisfying $\nabla e_i = \theta_i^j e_j$, since $\{e_i\}$ are all holomorphic sections, so from the *compatibility with the complex structure*, each θ_i^j is a smooth (1,0)-form.

Now we use the *compatibility with the metric* to get

$$\begin{aligned}\mathrm{d}h_{i\bar{j}} = \mathrm{d}h(e_i,e_j) &= h(\nabla e_i, e_j) + h(e_i,\nabla e_j)\\ &= h(\theta_i^k e_k, e_j) + h(e_i, \theta_j^k e_k)\\ &= \theta_i^k h_{k\bar{j}} + \bar{\theta}_j^k h_{i\bar{k}}\\ &\in \mathcal{A}^{(1,0)} + \mathcal{A}^{(0,1)},\end{aligned}$$

and since $\mathrm{d}h_{i\bar{j}} = \partial h_{i\bar{j}} + \bar{\partial} h_{i\bar{j}}$, comparing the types, we have $\partial h_{i\bar{j}} = \theta_i^k h_{k\bar{j}}$, so we vae

$$\boxed{\partial h = \theta^{\mathsf{T}} h} \quad \Rightarrow \quad \boxed{\theta^{\mathsf{T}} = \partial h \cdot h^{-1}},$$

Denote $h^{-1} = (h^{\bar{j}i})$, then we can rewrite this as

$$\theta_i^j = h^{\bar{k}j}\partial h_{i\bar{k}}. \tag{4.2}$$

Also, since $\bar{h}^{\mathsf{T}} = h$, the (0,1)-part gives the same equation, this proves the uniqueness.

(Part2) For existence, we simply set locally $\theta^i_j := h^{\bar{k}i}\partial h_{j\bar{k}}$ on U, and define for $s = s^i e_i$:

$$\nabla s := (\mathrm{d}s^i + s^j\theta^i_j)e_i,$$

and now we need to check that this is globally well-defined. For this, if $f_j = a^i_j e_i$ is another holomorphic frame on V with $U \cap V \neq \varnothing$. Then a is a holomorphic matrix. Furthermore, we have $\widetilde{h} = a^{\mathsf{T}} h \bar{a}$, this is because

$$\widetilde{h}_{i\bar{j}} = h(f_i, f_j) = a^k_i \bar{a}^l_j h_{k\bar{l}} = a^k_i h_{k\bar{l}} \bar{a}^l_j,$$

so we have the another connection forms are

$$\widetilde{\theta} := (\partial\widetilde{h} \cdot \widetilde{h}^{-1})^{\mathsf{T}} = a^{-1}\mathrm{d}a + a^{-1}\theta a,$$

since $s = \widetilde{s}^i f_i$, we have $s := (s^1, \cdots, s^r)^{\mathsf{T}}$, and $\widetilde{s} = (\widetilde{s}^1, \cdots, \widetilde{s}^r)^{\mathsf{T}}$ then $\widetilde{s} = a^{-1}s$, so

$$\begin{aligned}
&(\mathrm{d}\widetilde{s}^i + \widetilde{s}^j\widetilde{\theta}^i_j)f_i = f(\mathrm{d}\widetilde{s} + \widetilde{\theta}\widetilde{s}) \\
&= ea\left(\mathrm{d}(a^{-1}s) + (a^{-1}\mathrm{d}a + a^{-1}\theta a)(a^{-1}s)\right) \\
&= e(\mathrm{d}s + \theta s),
\end{aligned}$$

where $f = (f_1, \cdots, f_r)$, $e = (e_1, \cdots, e_r)$, so ∇ is globally defined.

Finally, we construct the unique Chern connection. ♣

Remark. If we define covariant derivatives of a smooth section s with respect to a complex tangent vector X at a given point p by $\nabla_X s := X(\nabla s) \in \mathcal{A}^0(E)$, where we use the dual pairing of tangent vectors and differential 1-forms. Then the *compatibility with metric* takes the form

$$\boxed{X(h(s,t)) = h(\nabla_X s, t) + h(s, \nabla_{\bar{X}} t)},$$

note that the second component is $\bar{X}$.

Remark. The line bundle case is particularly simple: if e is a local holomorphic frame and we set $h = h(e,e) > 0$, then the connection 1-form is $\theta = h^{-1}\partial h = \partial \log h$. Then the curvature $\Theta = \mathrm{d}\theta + \theta\wedge\theta = \mathrm{d}\theta = (\partial + \bar{\partial})(\partial \log h) = \bar{\partial}\partial \log h$, which is a globally defined closed (1, 1)-form.

Now we study the property of the curvature of Chen connection : In general, the curvature of Chern connection is locally given by

$$\Theta = \mathrm{d}\theta + \theta\wedge\theta = \bar{\partial}\theta + (\partial\theta + \theta\wedge\theta),$$

note that form the compatibility with the complex structure implies that θ is a (1,0)-form, so

$$\Theta = \Theta^{1,1} + \Theta^{2,0}, \quad \Theta^{1,1} = \bar{\partial}\theta, \quad \Theta^{2,0} = \partial\theta + \theta\wedge\theta.$$

However, an important obeservation is , where we use the local expression of θ,

$$
\begin{aligned}
\Theta^{2,0} &= \partial\theta + \theta\wedge\theta \\
&= \partial\left(H^{-1}\partial H\right) + (H^{-1}\partial H)\wedge(H^{-1}\partial H) \\
&= \partial H^{-1}\wedge\partial H + H^{-1}\partial^2 H + (H^{-1}\partial H)\wedge(H^{-1}\partial H) \\
&= -H^{-1}\partial H\cdot H^{-1}\wedge\partial H + (H^{-1}\partial H)\wedge(H^{-1}\partial H) \\
&= 0
\end{aligned}
$$

where $H = h^{\mathsf{T}}$, and we use the classical trick $\partial(H\cdot H^{-1}) = 0$ again, and $\partial^2 H = 0$.

So with respect to a local holomorphic frame we have $\Theta = \Theta^{1,1}$ is of type $(1,1)$, and locally

$$
\Theta = \bar{\partial}\left(H^{-1}\partial H\right), \quad H = h^{\mathsf{T}}.
$$

so we have the conclusion : For Chern connection on a holomorphic vector bundle, its curvature form is always of type (1,1) , regardless of whether the frame is holomorphic or not!

Now we calculate an example:

Example 4.2.1. *Consider the universal line bundle $\mathbb{U}\to\mathbb{CP}^n$, recall that*

$$
\mathbb{U} = \{([z],v)|v\in[z]\}\subset\mathbb{CP}^n\times\mathbb{C}^{n+1}.
$$

We can define a natural Hermitian metric on $\mathbb{U}$:

$$
h_{[z]}(v,w) := \langle v,w\rangle_{\mathbb{C}^{n+1}}.
$$

We now compute this metric and its curvature using local trivializations: Take $U_0 = \{[z]|z^0\neq 0\}$ for example, the coordinates are

$$
(\xi^1,\cdots,\xi^n) = \left(\frac{z^1}{z^0},\cdots,\frac{z^n}{z^0}\right),
$$

then then we can choose a local frame e, and $e([z]) := ([z],(1,\xi^1,\cdots,\xi^n))$, so we get

$$
h(e,e) = 1 + |\xi^1|^2 + \cdots + |\xi^n|^2,
$$

and hence

$$
\theta = \partial\log h = \frac{1}{1+|\xi^1|^2+\cdots+|\xi^n|^2}\sum_{i=1}^n 2\bar{\xi}^i\mathrm{d}\xi^i,
$$

and furthermore, if we denote $|\xi|^2 = |\xi^1|^2+\cdots+|\xi^n|^2$ for short, we have

$$
\boxed{\Theta = \bar{\partial}\partial\log h = -\left(\frac{\delta_{ij}}{1+|\xi|^2} - \frac{\bar{\xi}^i\xi^j}{1+|\xi|^2}\right)\mathrm{d}\xi^i\wedge\mathrm{d}\bar{\xi}^j}.
$$

4.3 Chern Classes of a Complex Vector Bundle

Before we introduce Chern classes, we need have a more precise understanding of curvature form Θ, actually, since $\Theta(f\xi) = f\Theta\xi$, so we can view Θ as a linear transformation, so it is actually a End$E-$valued 2-forms, and now we need to give a brief introduction of the bundle End$E$.

Definition 4.3.1

Let E be a complex vector bundle of rank r over X, **the bundle End E** is defined to be

$$\text{End } E := \bigsqcup_{p\in X} \text{End}_{\mathbb{C}}(E_p) = \bigsqcup_{p\in X} \text{Hom}_{\mathbb{C}}(E_p, E_p)$$

as a set. If we have a natural trivialization of E, $\pi^{-1}(U) \to U \times \mathbb{C}^r$, equivalently, we choose a local frame $\{e_i\}_{i=1}^r$. Now we have an induced local frame for End E:

$$\boxed{E_{ij} \in C^\infty(U, \text{End } E), \quad E_{ij}(e_k) = \delta_{jk}e_i}.$$

Under this frame, we get a trivialization map:

$$\widetilde{\pi}^{-1}(U) \to U \times M_r(\mathbb{C}),$$

where $M_r(\mathbb{C})$ is the linear space of $r\times r$ complex matrices. The trivialization map is given by

$$\left(p, \sum_{i,j} a_{ij}E_{ij}(p)\right) \mapsto (p, (a_ij)).$$

A local section $\sigma = \sum_{i,j} a_{ij}E_{ij} \in C^\infty(U, \text{End } E)$ can be identified with a $M_r(\mathbb{C})$ valued smooth function $A = (a_{ij}) : U \to M_r(\mathbb{C})$. So for a section $s = s^i e_i$, we have

$$\sigma s := \sigma(s) = \sum_{i,j} a_{ij}E_{ij}(s^k e_k) = \sum_{i,j} a_{ij}s^j e_i.$$

So under the trivilization, the action of σ on s is just the matrix (a_{ij}) times the column vector (s^k).

Now we change the local frame $\{e_\alpha\}$ to $\widetilde{e}_\alpha = a_\alpha^\beta e_\beta$, then we have a corresponding induced frame $\widetilde{E}_{\alpha\beta}$. Then for a local section $\sigma \in C^\infty(U, \text{End } E)$, suppose $(b_\alpha^\beta) = (a_\alpha^\beta)^{-1}$, if

$$\sigma = \sum_{\alpha,\beta} c_{\alpha\beta}E_{\alpha\beta} = \sum_{\alpha,\beta} \widetilde{c}_{\alpha\beta}\widetilde{E}_{\alpha\beta},$$

then we actually have

$$\sigma(e_\beta) = \sum_{\alpha,\beta} c_{\alpha\beta}e_\alpha, \quad \sigma(\widetilde{e}_\beta) = \sum_{\alpha,\beta} \widetilde{c}_{\alpha\beta}\widetilde{e}_\alpha.$$

so we get

$$\sum_{\alpha,\beta} \widetilde{c}_{\alpha\beta}\widetilde{e}_\alpha = \sigma\left(a^\gamma_\beta e_\gamma\right) = a^\gamma_\beta \sum_\mu c_{\mu\gamma} e_\mu = \sum_\mu a^\gamma_\beta c_{\mu\gamma} b^\alpha_\mu \widetilde{e}_\alpha,$$

and hence $\boxed{\widetilde{c} = a^{-1}ca}$.

So a smooth section of End E is given by a family of locally defined matrix-valued smooth functions $c_i : U_i \to M_r(\mathbb{C})$, and when $U_i \cap U_j \neq \varnothing$, we have

$$\boxed{c_i = \psi_{ij}^{-1} c_j \psi_{ij}}.$$

Similarly, End E-valued differential forms are locally given by

$$\eta = \sum_{i=1}^N \omega_i \otimes A_i,$$

where A_i is a matrix-valued smooth function and ω_i is a smooth $k-$form on a trivilization neighborhood U. To make it well-defined, we require that when we change the local frame by $\widetilde{e}_\alpha = a^\beta_\alpha e_\beta$, we have

$$\widetilde{\eta} = a^{-1}\eta a = \sum_{i=1}^N \omega_i \otimes (a^{-1}A_i a).$$

One now may feel confused about two definitions : matrix-valued form and matrix of diff forms , as above, a $M_r(\mathbb{C})$-valued differential form can always be written as a $r \times r$ matrix of differential forms: let $\eta = \sum_{i=1}^N \omega_i \otimes A_i$. Now suppose $A_i = (A^i_{\alpha\beta})$ with $A^i_{\alpha\beta}$ are smooth functions, then we have $A_i = \sum_{\alpha,\beta} A^i_{\alpha\beta} E_{\alpha\beta}$ and hence

$$\begin{aligned}\eta = \sum_i \omega_i \otimes A_i &= \sum_i \sum_{\alpha,\beta} \omega_i \otimes (A^i_{\alpha\beta} E_{\alpha\beta}) \\ &= \sum_{\alpha,\beta}\left(\sum_i \omega_i A^i_{\alpha\beta}\right) \otimes E_{\alpha\beta} \\ &=: \sum_{\alpha,\beta} \eta_{\alpha\beta} \otimes E_{\alpha\beta}.\end{aligned}$$

This means that we can view η as a matrix whose $(\alpha,\beta)-$entry is precisely the differential forms $\eta_{\alpha\beta} = \sum_i A^i_{\alpha\beta}\omega_i$. One may still feel confused, now we give an example

$$\mathrm{d}z_1 \otimes \begin{pmatrix} 1 & z_2^{2023} \\ 1 & 0 \end{pmatrix} + \mathrm{d}z_2 \otimes \begin{pmatrix} 0 & 1 \\ \mathrm{e}^{z_1} & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{d}z_1 & z_2^{2023}\mathrm{d}z_1 + \mathrm{d}z_2 \\ \mathrm{d}z_1 + \mathrm{e}^{z_1}\mathrm{d}z_2 & \mathrm{d}z_2 \end{pmatrix},$$

wher left hand is matrix-valued form, and the right hand is matrix of differential forms.

Now we introduce two operators on End E-valued differential forms , suppose $\eta = \eta_{\alpha\beta} \otimes E_{\alpha\beta}$, so we define the **trace** of η is $\mathrm{tr}\eta = \sum_\alpha \eta_{\alpha\alpha}$.

It is not hard to check the definition above equals to the definition below:

$$\text{tr } \eta = \sum_i (\text{tr } A_i)\omega_i.$$

Another tool we shall use is the **commutator**, defined by

$$\boxed{[\omega \otimes A, \eta \otimes B] := (\omega \wedge \eta) \otimes [A, B]},$$

this is defined using the matrix-valued forms, but we can see that it is really a commutator in linear algebra $[A, B] = AB - BA$, by using the matrix of differential forms, one can see that $\omega \otimes A = \omega A = (A_{\alpha\beta}\omega)$, and similarly $\eta \otimes B = (B_{\alpha\beta}\eta)$, so

$$[\omega \otimes A, \eta \otimes B] = \omega A \wedge \eta B - (-1)^{\deg(\omega)\deg(\eta)} \eta B \wedge \omega A.$$

One can easily check and we will not offer a proof here.

We sometimes extend the definition: we define for the connection ∇,

$$\boxed{[\nabla, \omega \otimes A]s := \nabla(\omega \otimes As) - (-1)^{\deg(\omega)} \omega \otimes A \wedge \nabla s}.$$

we generalize this definition is for the curvature form Θ .

Now we offer some technical lemmas:

Proposition 4.3.1

If ∇_1 and ∇_2 are two connections on E, then $\nabla_1 - \nabla_2 \in \mathcal{A}^1(\text{End } E)$.

Proof. For any $s \in C^\infty(E)$, and $f \in C^\infty$, we have

$$\begin{aligned}(\nabla_1 - \nabla_2)(fs) &= f(\nabla_1 - \nabla_2)s + \mathrm{d}fs - \mathrm{d}fs \\ &= f(\nabla_1 - \nabla_2)s,\end{aligned}$$

then we finish the proof. ♣

Proposition 4.3.2

If P, Q are both End E-valued differential forms, then $\text{tr}[P, Q] = 0$.

Proof. Suppose $P = \omega \otimes A$, and $Q = \eta \otimes B$, then $\text{tr}[P, Q] = \text{tr}(\omega \wedge \eta \otimes [A, B]) = \text{tr}([A, B])(\omega \wedge \eta) = 0$, then we finish the proof, general case is from the linearity of two operators. ♣

Proposition 4.3.3: Second Bianchi identity

We have $[\nabla . \Theta^k] = 0$, for any $k \in \mathbb{N}$.

Proof. Simply note that $\Theta = \nabla^2$, thus we have for any section s, $[\nabla, \nabla^{2k}]s = \nabla(\nabla^{2k})s - (-1)^{2k}\nabla^{2k}(\nabla s) = 0$, so we finish the proof easily. ♣

Remark. Now we pove that $[\nabla, \Theta] = 0$ is our familiar 2nd Bianchi identity. In fact, let $s = s^i e_i$ be a local section of E, then from

$$\begin{aligned} 0 = [\nabla, \Theta]s &= [\nabla, \Theta^i_j E^j_i](s^k e_k) \\ &= \nabla(\Theta^i_j s^j e_i) - \Theta^i_j E^j_i \wedge (\mathrm{d}s^k e_k + s^l \theta^k_l e_k) \\ &= [\mathrm{d}s^j \wedge \Theta^k_j + s^j \mathrm{d}\Theta^k_j + \Theta^i_j s^j \wedge \theta^k_i - \Theta^k_j \wedge (\mathrm{d}s^j + \theta^j_i s^i)]e_k \\ &= s^j[\mathrm{d}\Theta^k_j + \Theta^i_j \wedge \theta^k_i - \Theta^k_i \wedge \theta^i_j]e_k, \end{aligned}$$

since $\{s^j\}$ is arbitary, so we have $\mathrm{d}\Theta + \theta \wedge \Theta - \Theta \wedge \theta = 0$. In the Riemannian case,

$$\Theta^i_j = \frac{1}{2} R^i_{jpq} \mathrm{d}x^p \wedge \mathrm{d}x^q, \quad \theta^i_j = \Gamma^i_{jk} \mathrm{d}x^k,$$

thus we can easily have

$$\nabla_k R^i_{jpq} \mathrm{d}x^k \wedge \mathrm{d}x^p \wedge \mathrm{d}x^q = 0,$$

this is nothing but the more familiar formula

$$\boxed{\nabla_k R^i_{jpq} + \nabla_p R^i_{jqk} + \nabla_q R^i_{jkp} = 0}.$$

Proposition 4.3.4: The key lemma

For $A \in \mathcal{A}^k(\mathrm{End}\, E)$, we have $\mathrm{d}\,\mathrm{tr}(A) = \mathrm{tr}[\nabla, A]$.

Proof. First note that the left hand side is obviously independent of the connection. For the right hand side, if we use another connection ∇', then we have

$$\mathrm{tr}[\nabla', A] = \mathrm{tr}[\nabla' - \nabla, A] + \mathrm{tr}[\nabla, A] = \mathrm{tr}[\nabla, A],$$

where we use $\nabla' - \nabla \in \mathcal{A}^1(\mathrm{End}\, E)$, and $\mathrm{tr}[P, Q] = 0$.

So we can in fact choose a trivial conncetion locally to carry out the computation: let $\nabla_0 = \mathrm{d}$ be a trivial connection, where $\nabla_0 e_i = 0$ for the fixed frame, thus we have $\nabla_0 s^i e_i = \mathrm{d}s^i e_i$.Then

$$\begin{aligned} [\nabla_0, A]s &= \nabla_0(As) - (-1)^{\deg(A)} A \wedge \nabla_0 s \\ &= \nabla_0(A^i_j s^j e_i) - (-1)^{\deg(A)} A^i_j \wedge \mathrm{d}s^j e_i \\ &= \mathrm{d}(A^i_j s^j)e_i - (-1)^{\deg(A)} A^i_j \wedge \mathrm{d}s^j e_i \\ &= (\mathrm{d}A^i_j)s^j e_i = (\mathrm{d}A) \cdot s, \end{aligned}$$

so we have $\mathrm{tr}[\nabla_0, A] = \mathrm{tr}(\mathrm{d}A) = \mathrm{d}\,\mathrm{tr}(A)$, then we finish the proof. ♣

Now we have enough tools to talk about Chern-Weil theory : For any formal power series in one variable $f(x) = a_0 + a_1 x + \cdots$, we define

$$\boxed{f(\Theta) := a_0 + a_1\Theta + \cdots + a_n\Theta^n \in \mathcal{A}^*(X)},$$

where recall Θ is a 2-form, so $\Theta^k = 0$ for $k > n$.

Theorem 4.3.1: Chern-Weil Theorem

For f as above, we have

1. $\mathrm{d}\,\mathrm{tr} f(\Theta) = 0$;
2. If $\widetilde{\nabla}$ is another connection with curvature $\widetilde{\Theta}$, there is a differential form $\eta \in \mathrm{A}^*(X)$ such that $\mathrm{tr} f(\Theta) - \mathrm{tr} f(\widetilde{\Theta}) = \mathrm{d}\eta$.

So the cohomology class of $\mathrm{tr} f(\Theta)$ is independent of the connection. We call it the **characteristic class** of E associated to f, and $\mathrm{tr} f(\Theta)$ the corresponding **characteristic form** of E associated to f and ∇.

Proof. For the first conclusion, we have

$$\mathrm{d}\,\mathrm{tr} f(\Theta) = \mathrm{tr}[\nabla, f(\Theta)] = \sum_{k=1}^{n} a_k[\nabla, \Theta^k] = 0,$$

where we used Bianchi identity in the last step.

For the second conclusion, we choose a family of conncetions $\nabla_t := t\widetilde{\nabla} + (1-t)\nabla$, then

$$\dot{\nabla}_t := \frac{\mathrm{d}\nabla_t}{\mathrm{d}t} = \widetilde{\nabla}_t - \nabla \in \mathcal{A}^1(\mathrm{End}\ E),$$

so we actually have

$$\dot{\Theta}_t := \frac{\Theta_t}{\mathrm{d}t} = \frac{\mathrm{d}\nabla_t}{\mathrm{d}t} \wedge \nabla_t + \nabla_t \wedge \frac{\mathrm{d}\nabla_t}{\mathrm{d}t} = [\nabla_t, \dot{\nabla}_t].$$

Now we can change have the following :

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr} f(\Theta_t) &= \mathrm{tr}\left(\dot{\Theta}_t f'(\Theta_t)\right) = \mathrm{tr}\left([\nabla_t, \dot{\nabla}_t] f'(\Theta_t)\right) \\
&= \mathrm{tr}[\nabla_t, \dot{\nabla}_t f'(\Theta_t)] \qquad \text{(Bianchi)} \\
&= \mathrm{d}\,\mathrm{tr}\left(\dot{\nabla}_t f'(\Theta_t)\right),
\end{aligned}$$

so we can conclude that $\mathrm{tr} f(\Theta) - \mathrm{tr} f(\widetilde{\Theta}) = \mathrm{d}\int_0^1 \mathrm{tr}\left(\dot{\nabla}_t f'(\Theta_t)\right) \mathrm{d}t.$ ♣

Example 4.3.1 (Chern Class)**.** *Now* ***Chern class*** *is a special case of charactristic class, by choosing*

$$f(\Theta) := \det\left(I_r + \frac{\sqrt{-1}}{2\pi}\Theta\right) = \exp\left(\mathrm{tr}\ \log\left(I_r + \frac{\sqrt{-1}}{2\pi}\Theta\right)\right),$$

so by Taylor expansion, we have $f(\Theta) = 1 + c_1(E,\nabla) + \cdots + c_n(E,\nabla)$, where $c_i(E,\nabla) \in \mathcal{A}^{2i}(X)$ are called closed forms, whose cohomology class are all independent of ∇ from the theorem above. These are called ***Chern classes****, for example, from*

$$\exp(\text{tr } \log(I_r + A)) = \exp\left(\text{tr}\left(A - \frac{A^2}{2} + \frac{A^3}{3} + \cdots\right)\right)$$
$$= 1 + \text{tr}(A) + \left(\frac{1}{2}(\text{tr}A)^2 - \frac{1}{2}\text{tr}(A^2)\right) + \cdots,$$

wo we actually have

$$\boxed{c_1(E,\nabla) = \frac{\sqrt{-1}}{2\pi}\text{tr}\Theta, \quad c_2(E,\nabla) = \frac{1}{8\pi^2}\left((tr)(\Theta^2) - (\text{tr}\Theta)^2\right)}.$$

Now we offer more propositions about Chern classes :

Proposition 4.3.5

We can find a connction ∇ such that all chern forms $c_k(E,\nabla)$ are real.

Proof. Since the Chern classes are independent of the connection, so we can choose a metric h and require that ∇ is compatible with the metric. Choose a local unitary frame, so that $h_{i\bar{j}} = \delta_{ij}$, then

$$0 = \mathrm{d}h_{i\bar{j}} = \mathrm{d}h(e_i, e_j) = \theta_i^k \delta_{kj} + \delta_{ik}\bar{\theta}_j^k = \theta_j^i + \bar{\theta}_i^j,$$

in short, $\bar{\theta}^{\mathsf{T}} = -\theta$, this in turn implies that $\boxed{\bar{\Theta}^{\mathsf{T}} = -\Theta}$, and so $\overline{c(E,\nabla)} = c(E,\nabla)$. ♣

Now we shall prove that Chern classes are obstructions to the existence of global linearly independent smooth sections:

Theorem 4.3.2: Chern Classes are Obstructions

If $E \to X$ is a smooth complex vector bundle of rank r, if there are k smooth sections $s_1, \cdots, s_k \in C^\infty(E)$ such that $\{s_i(p)\}_{i=1}^k$ are linearly independent everywhere, then we have $c_i(E) > 0$ for $i > r - k$.

Proof. consider $E = T \oplus E'$ where T is trivial $k-$bundle. ♣

4.4 Hermitian Metrics and Kähler Metrics

Definition 4.4.1: Hermitian Metric

Let X be a complex manifold of dimension n, we denote the canonical almost complex structure by J. A Riemannian metric g on X is called **Hermitian**, if g is $J-$invariant, i.e.,

$$g(Ju, Jv) = g(u, v), \quad \forall u, v \in T_p^{\mathbb{R}} X, p \in X.$$

Remark. As before, we extend g to $T^{\mathbb{C}}X$ as a complex bilinear form. For simplicity, we also denote this bilinear form by g. Then we have

$$g(T^{1,0}, T^{1,0}) = g(JT^{1,0}, JT^{1,0}) = g(\sqrt{-1}T^{1,0}, \sqrt{-1}T^{1,0}) = -g(T^{1,0}, T^{1,0}),$$

so $g(T^{1,0}, T^{1,0}) = 0 = g(T^{0,1}, T^{0,1})$, then

$$\boxed{h(Z, W) := g(Z, \bar{W})}$$

defines an Hermitian metric on the rank n holomorphic vector bundle $T^{1,0}X = \text{span}\{\partial_{z^1}, \cdots, \partial_{z^n}\}$.

Definition 4.4.2: Kähler Form

For an Hermitian metric g on (X, J), we define the associated **Kähler form** ω_g by

$$\omega_g(u, v) := g(Ju, v).$$

Remark. Note that we have

$$\omega_g(u, v) = g(Ju, v) = g(J^2u, Jv) = -g(u, Jv) = -\omega_g(v, u),$$

so ω_g is a real 2-form on X.

Definition 4.4.3: Kähler Manifold

An Hermitian metric g on X is called **Kähler metric**, if $\mathrm{d}\omega_g = 0$. Its cohomology class in $H^2_{dR}(X)$ is called the **Kähler class** of g. If a complex manifold admits a Kähler metric, we call it the **Kähler manifold**.

Remark. Recall the definition of sympletic manifold, which is a $2n$ smooth manifold, with a nonwhere vanishing 2-form ω and $\mathrm{d}\omega = 0$, since we already have ω_g is a real 2-form, then by the definition of Kähler manifold, we know that Kähler manifold is also a sympletic manifold.

Locally, if $(z^1, \cdots, z^n)$ is a holomorphic coordinate system, then g is determined by $g_{i\bar{j}} := g(\partial_i, \partial_{\bar{j}})$, where $\partial_i = \dfrac{\partial}{\partial z^i}$, and $\partial_{\bar{j}} = \dfrac{\partial}{\partial \overline{z^j}}$, since $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Then we have

$$\boxed{\omega_g = \sqrt{-1} g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\overline{z^j}},$$

now we have

$$
\begin{aligned}
0 = \mathrm{d}\omega_g &= \sqrt{-1}\mathrm{d}g_{i\bar{j}} \wedge \mathrm{d}z^i \wedge \mathrm{d}\overline{z^j} \\
&= \sqrt{-1}\frac{\partial g_{i\bar{j}}}{\partial z^k}\mathrm{d}z^k \wedge \mathrm{d}z^i \wedge \mathrm{d}\overline{z^j} - \sqrt{-1}\frac{\partial g_{i\bar{j}}}{\partial \overline{z^l}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z^l} \wedge \mathrm{d}\overline{z^j} \\
&= \sqrt{-1}\left[\sum_j\sum_{k<i}\left(\partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}}\right)\mathrm{d}z^k \wedge \mathrm{d}z^i \wedge \mathrm{d}\overline{z^j} + \sum_i\sum_{j<l}\left(\partial_{\bar{l}} g_{i\bar{j}} - \partial_{\bar{j}} g_{i\bar{l}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z^j} \wedge \mathrm{d}\overline{z^l}\right)\right],
\end{aligned}
$$

so being Kähler mean that $g_{i\bar{j}}$ have the additional symmetries :

$$
\boxed{\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}, \quad \partial_{\bar{j}} g_{i\bar{l}} = \partial_{\bar{l}} g_{i\bar{j}},} \quad \forall i,j,k,l.
$$

Example 4.4.1. *The Euclidean metric $g = \sum_{i=1}^{n}(\mathrm{d}x^i \otimes \mathrm{d}x^i + \mathrm{d}y^i \otimes \mathrm{d}y^i)$ of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ is a Kähler metric, since we have*

$$
\partial_i = \frac{1}{2}\left(\partial_{x^i} - \sqrt{-1}\partial_{y^i}\right), \quad \partial_{\bar{j}} = \frac{1}{2}\left(\partial_{x^i} + \sqrt{-1}\partial_{y^i}\right),
$$

then we have $g_{i\bar{j}} = g(\partial_i, \partial_{\bar{j}}) = \frac{1}{2}\delta_{ij}$, so $\omega_g = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i$, since the coeffcients are all constans, then we know that $\mathrm{d}\omega_g = 0$.

To give more examples, note that to define a Kähler metrics, it suffices to define its associated Kähler form, since we have $g(u,v) = g(Ju,Jv) = \omega_g(u,Jv)$. So sometimes we will also say " Let ω_g be a Kähler metric ..."

Example 4.4.2. *Let $X = \mathbb{B}(1) \subset \mathbb{C}^n$ be the unit ball, we define a Kähler metric:*

$$
\omega_g := \sqrt{-1}\partial\bar{\partial}\log\frac{1}{1-|z|^2} = \sqrt{-1}g_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j,
$$

here we have $(g_{i\bar{j}}) = \left(\frac{\delta_{ij}}{1-|z|^2} + \frac{\bar{z}^i\bar{z}^j}{(1-|z|^2)^2}\right)$, which is positive definite, and $\mathrm{d}\omega_g = 0$ since $\mathrm{d}\partial\bar{\partial} = 0$, so it is indeed a Kähler metric, this is called the ***complex hyperbolic metric****.*

Example 4.4.3. *Let $X = \mathbb{CP}^n$ with homogeneous coordinates $[Z^0, \cdots, Z^n]$, we define a Kähler metric:*

$$
\omega_g := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(|Z^0|^2 + \cdots + |Z^n|^2\right),
$$

It is easy to check that this is well-defined. It is called the ***Fubini-Study metric****.*

Remark. However, not every compact complex manifold is Kähler, since, for example, $H^2_{dR}(X)$ must be non-trivial, for if not, $[\omega_g] = 0$ then will be exact, so $\int_X \omega_g^n = 0$ by Stokes theorem, but this is impossible since it is the multiple of the volume of X.

Theorem 4.4.1

Calabi-Eckmann manifolds, i.e., $S^{2p+1} \times S^{2q+1}$ are never Kähler, where $p, q \geq 1$.

Proof. By Kunneth formula, and $H^1(S^{2k+1}) = H^2(S^{2k+1}) = 0$, we know $H^2(X) = 0$. ♣

However, we still have a lot of Kähler manifolds:

Proposition 4.4.1

If X is Kähler and Y is a complex analytic submanifold of X, then Y is also Kähler.

Proof. Let g be a Kähler metric on X and $\iota : Y \to X$ be the embedding map, then $\iota^* g$ is a Kähler metric on Y and the associated Kähler form is just $\iota^* \omega_g$. ♣

Corollary 4.4.1

All projective algebraic manifolds are Kähler.

Proof. Recall projective algebraic manifolds are the complex submanifolds of $\mathbb{CP}^n$, then from Fubini-Study metric we know that $\mathbb{CP}^n$ is Kähler, so we finish the proof. ♣

In Riemannian geometry, normal coordinates are very useful in tensor calculations. The next theorem shows that being Kähler is both necessary and sufficient for the existence of complex analogue of normal coordinates.