Topics in Geometric Analysis

Lecture Notes Typed by Kai Zhu

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Abstract

This is a unfaithful note of the lectures of geometric analysis given by W.S Jiang, which is recorded by GXCMR
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Contents

1 Lecture 1: Gromov-Hausdorff Topology

Definition 1.1 (Metric Space)**.** *We call (X, d) is a metric space, if the metric d satisfies*

1. *$d(x, y) = d(y, x)$, for any $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;*
2. *triangle inequality, $d(x, y) + d(y, z) \geq d(x, z)$.*

Furthermore, we have

- *$Y \subset X$ is $\varepsilon-$dense, if $X \subset B_\varepsilon(Y) := \{x \in X : d(x, Y) < \varepsilon\}$, where $d(x, Y) := \inf_{y \in Y} d(x, y)$;*
- *$Y \subset X$ is a $\varepsilon-$net, if for arbitrary $y_1, y_2 \in Y$, $d(y_1, y_2) > \varepsilon$, and $X \subset B_\varepsilon(Y)$;*
- *X is totally bounded, if $\forall \varepsilon > 0$, there exists finite $\varepsilon-$net of X;*
- *(X, d) is compact if (X, d) is complete and totally bounded.*

Definition 1.2 (Hausdorff Distance)**.** *Let (X, d) be a complete metric space, $Z, W \subset X$ are bounded closed subsets, then we define the Hausdorff distance between G and H is*

$$d_H(Z, W) := \inf\{\varepsilon : Z \subset B_\varepsilon(W) \text{ and } W \subset B_\varepsilon(Z)\}. \tag{1.1}$$

Example 1.3. *$Z = [0, 1] \subset \mathbb{R}$, $W = [1, 2] \subset \mathbb{R}$, then we have $d_H(Z, W) = 1$.*

Theorem 1.4. *Let χ to be the all bounded closed subset of (X, d), then*

1. *If (X, d) is complete, then (χ, d_H) is also a complete metric space;*
2. *If (X, d) is compact, then (χ, d_H) is also a compact metric space;*

Proof. We only prove (χ, d_H) is a metric space: If $d_H(Z, W) = 0$, then we have $Z \subseteq W$ and $W \subseteq Z$, then we have $Z = W$, here we use the closed condition to show that $B_0(Z) = Z$. Then for the triangle inequality, for any $Z, W, Y \subset X$, assume $d_H(Z, W) = a$, $d_H(W, Y) = b$, then $\forall \varepsilon > 0$, we have

$$Z \subseteq B_{a+\varepsilon}(W), \quad W \subseteq B_{b+\varepsilon}(Y),$$

then from a basic observation: $B_\delta(B_\varepsilon(Y)) \subseteq B_{\delta+\varepsilon}(Y)$, we have $Z \subseteq B_{a+b+2\varepsilon}(Y)$, then similarly, $Y \subseteq B_{a+b+2\varepsilon}(Z)$, so we actually have

$$d_H(Z, Y) \leq a + b + 2\varepsilon = d_H(Z, W) + d_H(W, Y) + 2\varepsilon, \quad \forall \varepsilon > 0,$$

finally, we show that $d_H(Z, Y) \leq d_H(Z, W) + d_H(W, Y)$. Left part is for exercise. □

Definition 1.5 (Isometry). *If (X, d_X) and (Y, d_Y) are isometric metric spaces, if there exists $\phi : (X, d_X) \to (Y, d_Y)$ and $\psi : (Y, d_Y) \to (X, d_X)$ such that*

1. $d_X(x_1, x_2) = d_Y(\phi(x_1, \phi(x_2)))$, $d_Y(y_1, y_2) = d_X(\psi(y_1), \psi(y_2))$;
2. $\psi \circ \phi = \mathrm{Id}_X$, $\phi \circ \psi = \mathrm{Id}_Y$.

Definition 1.6 (Isometric Embedding). *we denote $(X, d_X) \overset{\text{iso}}{\hookrightarrow} (Y, d_Y)$, if there exists $\phi : (X, d_X) \to (Y, d_Y)$ such that ϕ is isometry and injective.*

Definition 1.7 (Gromov-Hausdorff Distance). *If (X, d_X) and (Y, d_Y) are two compact metric spaces, then we define the Gromov-Hausdorff distance of X, Y is*

$$d_{GH}(X, Y) := \inf_{(Z, d_Z)} \left\{ d_H^Z(X, Y) \middle| X \overset{\text{iso}}{\hookrightarrow} Z, Y \overset{\text{iso}}{\hookrightarrow} Z \right\}. \tag{1.2}$$

Remark 1.8. *The definition above is well-defined, we assume* $\mathrm{diam}(X, d_X), \mathrm{diam}(Y, d_Y) \le D$, *then let $Z = X \sqcup Y$, and the metric d_Z on Z is given by $d_Z|_X = d_X$, $d_Z|_Y = d_Y$, and $d_Z(x, y) = D$.*

Actually, there is no need to consider genneral space Z other than $X \sqcup Y$, we call d is an admissible metric, if $d|_X = d_X$, $d|_Y = d_Y$, and d is a metric on $X \sqcup Y$, then we actually have

$$d'_{GH}(X, Y) := \inf_{(X \sqcup Y, d), d \text{ is admissible}} \{d_H(X, Y)\} = d_{GH}(X, Y). \tag{1.3}$$

Proof. By definition, we have $d'_{GH}(X, Y) \ge d_{GH}(X, Y)$. On the other hand, for any $\varepsilon > 0$, there exists (Z, d_Z) and $X \overset{\text{iso}}{\hookrightarrow} Z, Y \overset{\text{iso}}{\hookrightarrow} Z$, such that $d_H^Z(X, Y) \le d_{GH}(X, Y) + \varepsilon$. Now consider the product space $(Z \times [0, \varepsilon], d'_Z)$ with product metric, then naturally we have $(X, d_X) \overset{\text{iso}}{\hookrightarrow} (Z \times \{0\}, d'_Z)$, and $(Y, d_Y) \overset{\text{iso}}{\hookrightarrow} (Z \times \{\varepsilon\}, d'_Z)$.

Now let $(X \sqcup Y, d')$ is the restriction from $X \times [0, \varepsilon]$, then by definition, we have $d'_{GH}(X, Y) \le d_H^{d'}(X, Y)$, and

$$\begin{aligned} d_H^{d'}(X, Y) &= d_H^{d'}(X \times \{0\}, Y \times \{\varepsilon\}) \\ &\le d_H^{d'}(X \times \{0\}, X \times \{\varepsilon\}) + d_H^{d'}(X \times \{\varepsilon\}, Y \times \{\varepsilon\}) \\ &\le \varepsilon + d_H^Z(X, Y) \le d_{GH}(X, Y) + 2\varepsilon, \end{aligned}$$

so we have $d'_{GH}(X, Y) \le d_{GH}(X, Y)$, then actually $d_{GH}(X, Y) = d'_{GH}(X, Y)$, from this we finish the proof. □

Definition 1.9. *We Denote $\mathscr{M} = \{$all compact metric spaces$\}/$isometric class.*

We will show $(\mathscr{M}, d_{GH})$ is a complete metric space, this needs several steps:

Lemma 1.10. *For $(\mathscr{M}, d_{GH})$, it satisfies triangle inequality.*

Proof. For triangle inequality, for arbitrary compact spaces $(X, d_X), (Y, d_Y)$ and (Z, d_Z), our goal is $d_{GH}(X, Y) + d_{GH}(Y, Z) \ge d_{GH}(X, Z)$, assume $d_{GH}(X, Y) = a$, $d_{GH}(Y, Z) = b$, then there exisits admissible metric d_{XY} on $X \sqcup Y$ and d_{YZ} on $Y \sqcup Z$, such that

$$d_H^{d_{XY}}(X, Y) \le a + \varepsilon, \quad d_H^{d_{YZ}}(Y, Z) \le b + \varepsilon,$$

now define admissible metric d_{XW} on $X \sqcup W$:

$$d_{XZ}(x, z) := \inf_{y \in Y} \{d_{XY}(x, y) + d_{YZ}(y, z)\}, \tag{1.4}$$

then one can check this is well defined. Now define a metric d on $W = X \sqcup Y \sqcup Z$ such that

$$d|_{X \sqcup Y} = d_{XY}, \quad d|_{Y \sqcup Z} = d_{YZ}, \quad d|_{X \sqcup Z} = d_{XZ}, \tag{1.5}$$

one can also check this is a metric , now in the total space W, we have

$$d_H^W(X, Z) \le d_H^W(X, Y) + d_H^W(Y, Z) = d_H^{d_{XY}}(X, Y) + d_H^{d_{YZ}}(Y, Z) \le a + \varepsilon + b + \varepsilon,$$

so we have $d_{GH}(X, Z) \le d_H^W(X, Z) \le a + b = d_{GH}(X, Y) + d_{GH}(Y, Z)$. □

The several lemmas given below will offer some tools to estimate the Gromov-Hausdorff distance:

Lemma 1.11. *If $(Y, d_X) \subseteq (X, d_X)$ is $\varepsilon-$dense, then $d_{GH}(X, Y) \le \varepsilon$.*

Lemma 1.12. *Let $(X = \{x_1, \cdots, x_N\}, d_X)$ and $(Y = \{y_1, \cdots, y_N\}, d_Y)$, $d_X(x_i, x_j) = d_{ij}$, $d_Y(y_i, y_j) = h_{ij}$, assume*

$$|d_{ij} - h_{ij}| < \varepsilon, \quad \forall i.j,$$

then $d_{GH}(X, Y) \le \varepsilon$.

Proof. Let $Z = X \sqcup Y$, define admissble metric d_Z such that

$$d_Z(x_i, y_j) = \varepsilon + \min_k \{d_{ik} + h_{kj}\}, \quad d_Z(x_i, y_i) = \varepsilon,$$

then d_Z is a metric and (Check!). Thus, $d_H^Z(X, Y) = \varepsilon$, thus $d_{GH}(X, Y) \le \varepsilon$. □

Definition 1.13 ($\varepsilon-$GH Map)**.** *We call $f : X \to Y$ is $\varepsilon-GH$ map, if it satisfies*

1. *$\varepsilon-$isometry, i.e., $|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon$;*
2. *$\varepsilon-$onto, i.e., $Y \subseteq B_\varepsilon(f(X))$.*

Remark 1.14. *$0-GH$ map is actualy a isometry.*

Theorem 1.15. *we have the following dual statements:*

1. *If $f : X \to Y$ is a $\varepsilon-GH$ map, then $d_{GH}(X, Y) \le 6\varepsilon$;*
2. *If $d_{GH}(X, Y) < \varepsilon$, then there exists $6\varepsilon-GH$ map $f : X \to Y$.*

Proof. The main idea is using discrete points to subsitute the compact sets.

(Part 1) For the first statement, choose $X_\varepsilon = \{x_1, \cdots, x_N\} \subset X$ is a $\varepsilon-$net, then we have $X \subset B_\varepsilon(X_\varepsilon)$, since for all $x \in X$, then $\exists x_i \in X_\varepsilon$, such that $d_X(x, x_i) < \varepsilon$. Then from definition 1.13, we have $d_Y(f(x), f(x_i)) \le d_X(x, x_i) + \varepsilon \le 2\varepsilon$, so $f(X) \subset B_{2\varepsilon}(f(X_\varepsilon))$, then $Y \subset B_\varepsilon(f(X)) \subset B_{3\varepsilon}(f(X_\varepsilon))$. Now consider X_ε and $f(X_\varepsilon)$, by definition 1.13 and lemma 1.12, we have $d_{GH}(X_\varepsilon, f(X_\varepsilon)) \le \varepsilon$. So now from triangle inequality, we have

$$d_{GH}(X, Y) \le d_{GH}(X, X_\varepsilon) + d_{GH}(X_\varepsilon, f(X_\varepsilon)) + d_{GH}(f(X_\varepsilon), Y) \le \varepsilon + \varepsilon + 2\varepsilon = 5\varepsilon.$$

(Part 2) For the second statement, let $X_\varepsilon = \{x_1, x_2, \cdots, x_N\} \subseteq X$ be a $\varepsilon-$net, then we claim: there exists $Y_\varepsilon = \{y_1, \cdots, y_N\}$, such that "$d(x_i, y_i) \le \frac{3\varepsilon}{2}$".

Now we prove the claim above: since $d_{GH}(X, Y) \le \varepsilon$, then there exists (Z, d_Z) such that $X \overset{\text{iso}}{\hookrightarrow} Z, Y \overset{\text{iso}}{\hookrightarrow} Z$, and $d_H^Z(X, Y) \le \frac{5\varepsilon}{4}$, which means that $X \subset B_{3\varepsilon/2}(Y)$, then $\exists Y_\varepsilon = \{y_1, \cdots, y_N\}$ such that $d_Z(x_i, y_i) \le \frac{3\varepsilon}{2}$, then

$$|d_X(x_i, x_j) - d_Y(y_i, y_j)| \le d_Z(x_i, y_i) + d_Z(x_j, y_j) \le 3\varepsilon. \tag{1.6}$$

Another claim: $Y \subset B_{4\varepsilon}(Y_\varepsilon)$, note that from $Y \subset B_{3\varepsilon/2}(Y)$, for $\forall y \in Y$, $\exists x \in X$ such that $d_Z(x, y) \le \frac{3\varepsilon}{2}$, since X_ε is a net, then $\exists x_i \in X_\varepsilon$ such that $d_Z(x, x_i) \le \varepsilon$, then we have

$$d_Z(y, y_i) \le d_Z(y, x) + d_Z(x, x_i) + d_Z(x_i, y_i) \le \frac{3\varepsilon}{2} + \varepsilon + \frac{3\varepsilon}{2} = 4\varepsilon,$$

this shows that $Y \subset B_{4\varepsilon}(Y_\varepsilon)$, which is our second claim.

Now we define $f : X \to Y_\varepsilon \subset Y$, such that $f(x_i) = y_i$ and if $x \notin X_\varepsilon$, then there exists $x_i \in X_\varepsilon$ such that $d_Z(x, x_i) = \min_{1 \le j \le N} d_Z(x, x_j) \le \varepsilon$, then $f(x) = y_i$ (if x_i is the only choice, then just choose one as you like), geometrically, one can view we almost send the points "parallelly" from X to Y, then from (1.6), one can check $6\varepsilon-$isometry, and since $f(X) = Y_\varepsilon$, so $6\varepsilon-$onto is also trivial. □

Remark 1.16. *When we esitimate $d_H(U, V)$, naturally when $U \subset B_\varepsilon(V)$, and $V \subset B_\varepsilon(U)$, then $d_H(U, V) \le \varepsilon$, note that in most cases, we naturally may have $U \subset V = B_0(V) \subset B_\varepsilon(Y)$, so we only need one side estimation, this process appears but omits in the proof above.*

Remark 1.17. *From theorem 1.15, we know that the existence of $\varepsilon-$map is equivalent to the Gromov-Hausdorff distance is small.*

Now recall our goal:

Lemma 1.18. *For $(\mathscr{M}, d_{GH})$, if $d_{GH}(X,Y)=0$, then X is isometric to Y.*

Proof. This is a direct corollary from remark 1.14 and 1.17. □

From lemma 1.10 and 1.18, we have:

Theorem 1.19. *$(\mathscr{M}, d_{GH})$ is a complete and separable metric space.*

Remark 1.20. *A necessary reference is the chapter 11 of [1].*

Definition 1.21 (Fuctions are closed in GH-sense)**.** *Suppose (X,d_X) and (Y,d_Y) are two compact metric spaces with $d_{GH}(X,Y)\le\varepsilon$, and $f:X\to\mathbb{R}$, $g:Y\to\mathbb{R}$ are two functions, we say $|g-h|\le\varepsilon$ in GH-sense, if there exists $\varepsilon-$map $h:X\to Y$ such that*

$$\sup_{x\in X}|f(x)-g\circ h(x)|\le\varepsilon. \tag{1.7}$$

Example 1.22. *Let $f,g:[0,1]\to\mathbb{R}$, and $\Gamma_f,\Gamma_g\subset\mathbb{R}^2$ are the images of f,g, then trivially, we have*

$$d_{GH}(\Gamma_f,\Gamma_g)=d_H^{\mathbb{R}^2}(\Gamma_f,\Gamma_g)\le\max_{x\in[0,1]}|f(x)-g(x)|,$$

one can see this by drawing a picture and note that in this case , d_{GH} is similar to the L^∞ norm.

Example 1.23. *Let $f,g:[0,1]\to\mathbb{R}$ and continuous differentiable, if $\Gamma_f,\Gamma_g\subset\mathbb{R}^2$ are the images of f,g, and let $d_{\Gamma_f}(x_1,x_2)=$the length connecting x_1 and x_2, $d_{\Gamma_g}(y_1,y_2)=$the length connecting y_1 and y_2, then maybe non-trivially, there is a constant C only depends on f,g such that*

$$d_{GH}(\Gamma_f,\Gamma_g)\le C\left(\max_{x\in[0,1]}|f(x)-g(x)|,\max_{x\in[0,1]}|f'(x)-g'(x)|\right),$$

note that here we use the intrinsic metric of Γ_f and Γ_g, so the case tends to be more difficult.

From the examples above, we have two interesting computations:

Exercise 1.24. *Compute $d_H^{\mathbb{R}^2}(\mathbb{S}^1_r(0),\mathbb{S}^1_R(0))$ and $d_{GH}(\mathbb{S}^1_r(0),\mathbb{S}^1_R(0))$, where the second we use intrinsic metric as example 1.23.*

Definition 1.25 (Path and Length)**.** *Let (X,d) be a path connected metric space, then for each curve $\gamma:[0,1]\to X$, we define the lengh of γ is*

$$L[\gamma]:=\sup_{0=t_0<t_1<\cdots<t_N=1}\left(\sum_{i=0}^{N-1}d\left(\gamma(t_i),\gamma(t_{i+1})\right)\right). \tag{1.8}$$

Definition 1.26 (Length Space)**.** *A metric space (X,d) is called a length space, if it is path connected, and for all $x_0,x_1\in X$, there exists a path (or preisely, geodesic) γ such that $\gamma(0)=x_0$, $\gamma(1)=x_1$ and $L[\gamma]=d(x_1,x_2)$.*

The most important results is

Theorem 1.27. *If $\{(X_i,d_i)\}_{i\in\mathbb{N}}$ be compact length spaces, and converges to (X,d) under Gromov-Hausdorff distance, then (X,d) is also a length space.*

2 Lecture 2: Gromov's Precompactness Theorem

We prove theorem 1.27 firstly:

Lemma 2.1. *Let (X,d) is complete, then X is a length space if and only if for all $x_1,x_2\in X$, there exists a midpoint x_3 of x_1,x_2, i.e., $d(x_1,x_3)=d(x_3,x_2)=d(x_1,x_2)/2$.*

Proof. On the one hand is trivial, if X is a length space, then there exists a minimal curve γ connecting x_0 and x_1, then choose t_0 such that $d(x_1,x_3)=d(x_1,x_2)/2$, then note that $d(x_1,x_3)+d(x_3,x_2)\le L[\gamma]=d(x_1,x_2)\le d(x_1,x_3)+d(x_2,x_3)$, thus they all equal, then $d(x_1,x_3)=d(x_3,x_2)=d(x_1,x_2)/2$.

On the other hand, from the midpoint condition, we know that we can find $x_{1/2}$ for x_0 and x_1, then by induction for $\forall t\in\left\{\frac{i}{2^k}|0\le i\le 2^k,k\ge1\right\}:=T$, we have $x_{t_1+t_2/2}$ is the midpoint of x_{t_1} and x_{t_2}, now we can define $\gamma:T\to X$, by easily check, we have $d(x_t,x_s)=|t-s|d(x_0,x_1)$, since $T\subset[0,1]$ is dense, so we can extend to $[0,1]$, then is the desired minimal curve. □

Proof of themrem 1.27. Easily see that (X,d) is complete(Check!), then from lemma 2.1, it suffices to check midpoint condition, now if $x_1 = \lim_{i\to\infty} x_1^{(i)}$, and $x_2 = \lim_{i\to\infty} x_2^{(i)}$, and since each X_i is length space, then there exists $x_{1/2}^{(i)}$ is the midpoint of $x_1^{(i)}$ and $x_2^{(i)}$, then we can know that $x_{1/2} = \lim_{i\to\infty} x_{1/2}^{(i)}$ is the midpoint of $x_1^{(i)}$ and $x_2^{(i)}$(Check!), so we know that (X,d) is also a length space. □

Definition 2.2 (PGH-Distance)**.** *Let (X,d_X,x) and (Y,d_Y,y) are two compact pointed metric space, then we define the PGH-distance is*

$$d_{GH}^P((X,x),(Y,y)) := \inf_{(Z,d_Z)} \left\{ d_H^Z(X,Y) + d_Z(x,y) \middle| X \overset{\text{iso}}{\hookrightarrow} Z, Y \overset{\text{iso}}{\hookrightarrow} Z \right\}. \tag{2.1}$$

Example 2.3. *Let $(X,x) = ([0,2],0) \subset \mathbb{R}$, and $(Y,y) = ([0,2],1) \subset \mathbb{R}$, then $d_{GH}^P = 1$.*

Definition 2.4 (PGH-Convergence)**.** *We say (X_i,d_i,x_i) converge to (X,d,x) with PGH-distnace, if for all $R>0$, $(\bar{B}_R(x_i),d_i,x_i) \overset{d_{GH}^P}{\to} (\bar{B}_R(x),d,x)$.*

I have some questions about this definition.

Definition 2.5 (Capacity)**.** *Let (X,d) is a compact metric space, $r>0$, we define*

$$\mathrm{Cap}_X(r) := \max \# \left\{ B_{\frac{r}{2}}(x) \subset X \text{ and pairwise disjoint} \right\}, \tag{2.2}$$

then since X is compact, we know that $\mathrm{Cap}_X(r) < \infty$ for all $r>0$.

Remark 2.6. *If $\mathrm{Cap}_X(r) \le N$, then $\exists N$ many B_r-balls cover X, one can prove it by contradiction.*

Lemma 2.7. *If $d_{GH}(X,Y) \le \frac{\varepsilon}{3}$, then for all $r>0$, $\mathrm{Cap}_X(r+\varepsilon) \le \mathrm{Cap}_Y(r)$.*

Proof. Assume $\mathrm{Cap}_X(r+\varepsilon) = N$, then there exists N pairwise disjoint balls $B_{r+\varepsilon/2}(x_i)$ for $i=1,\cdots,N$, then $d(x_i,x_j) \ge r+\varepsilon$, since $d_{GH}(X,Y) \le \frac{\varepsilon}{3}$, then there exists $\{y_1,\cdots,y_N\} \subset Y$ such that $d(x_i,y_i) \le \frac{\varepsilon}{2}$, then we have $d(y_i,y_j) \ge d_(x_i,x_j) - d(x_i,y_i) - d(x_j,y_j) \ge r$, then we have $B_{r/2}(y_i) \cap B_{r/2}(y_j) = \varnothing$, thus we have $\mathrm{Cap}_Y(r) \ge N$. □

Theorem 2.8 (Gromov Precompactness Theorem)**.** *Let $\mathscr{C}$ be a collection of compact metric spaces, then $\mathscr{C} \subset (\mathscr{M}, d_{GH})$ is precompact(each sequence has a convergent subsequence) if and only if there exists a map $N:(0,1)\to\mathbb{Z}_+$ such that $\mathrm{Cap}_X(r) \le N(r)$ for all $X \in \mathscr{C}$ and $0<r<1$.*

Proof. Asuume $\mathscr{C}$ is precompact then equivalently, it is totally bounded, then for all $\varepsilon>0$, there exists finite $\{X_1,\cdots,X_{k_\varepsilon}\} \subset \mathscr{C}$, such that for all $X \in \mathscr{C}$, there exits $X_i \in \{X_1,\cdots,X_{k_\varepsilon}\}$ such that $d_{GH}(X_i,X) \le \frac{\varepsilon}{30}$, set $N(\varepsilon) = \max_{1\le i\le k_\varepsilon} \mathrm{Cap}_{X_i}(\varepsilon/10) < \infty$, then by lemma 2.7, we have $\mathrm{Cap}_X(\frac{9}{10}\varepsilon + \frac{1}{10}\varepsilon) \le \mathrm{Cap}_{X_i}(\frac{9}{10}\varepsilon) \le N(\varepsilon)$.

On the other hand, if for $\forall\varepsilon>0$ and $\forall X \in \mathscr{C}$, since $\mathrm{Cap}_X(\varepsilon) \le N(\varepsilon)$, then there exists $X^\varepsilon := \{x_X^1,\cdots,x_X^{N(\varepsilon)}\} \subset X$ such that $X \subset B_\varepsilon(\{x_X^1,\cdots,x_X^{N(\varepsilon)}\})$, then $d_{GH}(X,X^\varepsilon) \le \varepsilon$, and $\mathrm{diam}X \le 2\varepsilon \cdot N(\varepsilon)$, now let $f:\mathscr{C} \to \mathbb{R}^{N(\varepsilon)(N(\varepsilon)-1)/2}$, and $X \mapsto (e_{ij}(X))$, where $e_{ij}(X) = d_X(x_X^i,x_X^j)$, then $f(\mathscr{C}) \subset \mathbb{R}^{\cdots}$ is a bounded subset, the left part one can refer [1]. □

Definition 2.9 (Doubly Measure)**.** *If (X,d,μ) is a compact metric measure space, μ is a Borel measure, and $0<\mu(X)<+\infty$, we say μ is doubly measure with $\kappa>0$, if*

$$\mu(B_r(x)) \ge \kappa\mu(B_{2r}(x)), \quad \forall B_r(x) \subset X. \tag{2.3}$$

Lemma 2.10. *Let (X,d,μ) is a doubly measure space with $\kappa>0$, assume $\mathrm{diam}X \le D$, then for all $0<r\le D$,*

$$\mathrm{Cap}_X(r) \le C(\kappa) \left(\frac{D}{r}\right)^{\alpha(\kappa)},$$

where $C(\kappa), \alpha(\kappa) > 0$.

Proof. For all $x \in X$, we have $\mu(B_{r/2}(x)) \ge \kappa^{N+1}\mu(B_{2^N r}(x))$, where $D \le 2^N r < 2D$, then we have $N-1 \le \log_2 \frac{D}{r}$, then we have $\kappa^{N+1} \ge \kappa^2 \cdot \kappa^{\log_2 \frac{D}{r}} = \kappa^2 \cdot \left(\frac{D}{r}\right)^{\log_2 k}$,then we have

$$\mu\left(B_{r/2}(x)\right) \ge \kappa^2 \cdot \left(\frac{D}{r}\right)^{\log_2 k} \cdot \mu(X),$$

then we have

$$\mathrm{Cap}_X(r) \le \frac{\mu(X)}{\mu\left(B_{r/2}(x)\right)} \le \kappa^{-2} \cdot \left(\frac{D}{r}\right)^{\log_k 2},$$

then we finish the proof. ☐

Remark 2.11. *The doubly measure condition comes from the comparison of volume theorem, so it is general in Riemmanian geometry. So this lemma helps us to control capcity, then we can study GH-convergence.*

Now we turn to some Riemmanian manifolds with lower Ricci bound:

Definition 2.12 (Curvature Tensor)**.** *For $X, Y, Z, W \in \mathfrak{X}(M)$, we define*

$$\langle R(X,Y)Z, W\rangle := \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\rangle. \tag{2.4}$$

Definition 2.13 (Ricci Curvature)**.** *We define Ricci curvature of X, Y is*

$$\mathrm{Ric}(X,Y) := \mathrm{tr}\langle R(\cdot, X)Y, \cdot\rangle = \mathrm{Ric}(Y,X). \tag{2.5}$$

Definition 2.14 (Gradient,Hessian and Laplacian)**.** *Let $x \in M$, and $(U, x^1, \cdots, x^n)$ is the local coordinate of x, then if $u : M \to \mathbb{R}$ is a smooth function, then we define ∇u as*

$$\langle \nabla u, X\rangle := X(u), \tag{2.6}$$

then more precisely, we have

$$\nabla u = g^{ij} \partial_i u \partial_j. \tag{2.7}$$

We define Hess$u$ *or $\nabla^2 u$ is a $(0,2)-$tensor, such that*

$$\nabla^2 u(X,Y) := \langle \nabla_X \nabla u, Y\rangle, \tag{2.8}$$

locally, we have

$$\nabla^2 u(\partial_i, \partial_j) = \partial_i \partial_j u - \Gamma_{ij}^k \partial_k u. \tag{2.9}$$

We also have the Laplacian of u

$$\Delta u = \mathrm{tr}(\nabla^2 u) = g^{ij} \nabla^2 u(\partial_i, \partial_j). \tag{2.10}$$

Theorem 2.15 (Bochner Formula)**.** *If $u \in C^\infty(M)$, then we have*

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{2.11}$$

Proof. Fix $x \in M$, choose normal coordinate $(U, x^1, \cdots, x^n)$ such that $\partial_k g_{ij}(x) = 0$, and $g_{ij}(x) = \delta_{ij}$, then we have

$$\nabla_{\partial_i} \partial_j|_x = \Gamma_{ij}^k(x) \partial_k|_x = 0, \tag{2.12}$$

so we have at x, ☐

References

[1] P. Petersen, *Riemannian geometry.* Springer, 2006, vol. 171.