1st Geometry and Topology Summer School

Differential Forms in AT

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Summary

Rievew of Last seminar

Some Basic Notations

1. Ω^*: The algebra over $\mathbb{R}$ generated by $\{\mathrm{d}x^i\}_{1\leq i\leq n}$;
2. C^∞ differential forms:

$$\Omega^*(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \bigotimes_{\mathbb{R}} \Omega^*;$$

3. differential operator d: $\Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$, with
   - if $f \in \Omega^0(\mathbb{R}^n)$, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\mathrm{d}x^i =: \frac{\partial f}{\partial x^i}\mathrm{d}x^i;$$

   - if $\omega = f_I \mathrm{d}x^I$, then $\mathrm{d}\omega = \mathrm{d}f_I \wedge \mathrm{d}x^I$.

Some Basic Notations

4 de Rham complex on $\mathbb{R}^n$:

$$\{\Omega^*(\mathbb{R}^n), \mathrm{d}\} : 0 \to \Omega^0(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \Omega^1(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \Omega^2(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \cdots;$$

5 de Rham cohomology of $\mathbb{R}^n$:

$$H^*_{DR}(\mathbb{R}^n) := \mathrm{Kerd}/\mathrm{Imd};$$

6 $H^q(\mathbb{R}^1) = \begin{cases} \mathbb{R} & \text{if } q = 0 \\ 0 & \text{if } q \geq 1 \end{cases}$, we will soon show

$$\text{(Poincaré lemma)} \quad H^q(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } q = 0 \\ 0 & \text{if } q \geq 1 \end{cases}.$$

Some Important results

Theorem (short exact seq induces long exact seq)

Given a short exact seq of differential complexes:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

there is a long exact seq of cohomological groups :

$$\cdots \to H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C) \to H^{q+1}(A) \to \cdots .$$

Some Important results

Theorem (compact cohomology)

It is not hard to see

$$H_c^*(\mathrm{pt}) = \begin{cases} \mathbb{R} & \textit{in dimension } 0 \\ 0 & \textit{otherwise} \end{cases}, H_c^*(\mathbb{R}) = \begin{cases} \mathbb{R} & \textit{in dimension } 1 \\ 0 & \textit{otherwise} \end{cases},$$

we will soon see

$$(\textit{Poincaré lemma}) \quad H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \textit{in dimension } n \\ 0 & \textit{otherwise} \end{cases}.$$

Mayer-Vietoris Sequence

Theorem

Suppose $M = U \cup V$ with U, V open, then the Mayer-Vietoris seq below is exact:

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0,$$

and

$$0 \leftarrow \Omega_c^*(M) \leftarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \leftarrow \Omega_c^*(U \cap V) \leftarrow 0,$$

What we know? $H^*(\mathbb{R}^1)$

step by step

induction.

What we want? $H^*(\mathbb{R}^n)$

Poincaré Lemmas

Goal: Find relation between

$H^*(\mathbb{R}^{n+1})$ and $H^*(\mathbb{R}^n)$

The Poincaré lemma for de Rham cohomology

In this section, our main goal is to compute the ordinary cohomology and the compactly supported cohomology of $\mathbb{R}^n$, a baisic but improtant observation is

$$\begin{tikzcd}
\mathbb{R}^n \times \mathbb{R}^1 \arrow[d, shift right, "\pi"'] & \arrow[r, "\Omega^*"] & & \Omega^*(\mathbb{R}^n \times \mathbb{R}^1) \arrow[d, shift left, "s^*"] & \arrow[r, squiggly] & & \boxed{H^*(\mathbb{R}^n \times \mathbb{R}^1)} \arrow[d, shift left, "s^\#"] \\
\mathbb{R}^n \arrow[u, shift right, "s"'] & & & \Omega^*(\mathbb{R}^n) \arrow[u, shift left, "\pi^*"] & & & H^*(\mathbb{R}^n) \arrow[u, shift left, "\pi^\#"]
\end{tikzcd}$$

$\pi(x,t) = x$. $s(x) = (x,0)$ $\quad s^*f(x) = f(x,0)$

We will show that these maps induce inverse isomorphisms in cohomology and therefore $H^*(\mathbb{R}^{n+1}) \cong H^*(\mathbb{R}^n)$.

The Poincaré lemma for de Rham cohomology

$\Rightarrow s^\# \circ \pi^\# = \mathrm{id}$

$\pi^\# \circ s^\#$

Since $s^* \circ \pi^* = (\pi \circ s)^* = \mathrm{id}$, but the other side doesn't hold, but we hope there exists K such that

$$1 - \pi^* \circ s^* = \pm(\mathrm{d}K \pm K\mathrm{d}),$$

$\pi^* \circ s^*$

$K: \Omega^*(\mathbb{R}^n \times \mathbb{R}) \longrightarrow \Omega^{*-1}(\mathbb{R}^n \times \mathbb{R})$

Check! : if we have K. then $\pi^\# \circ s^\# = \mathrm{id}$.

The Poincaré lemma for de Rham cohomology

$K: \underline{\Omega^q}(\mathbb{R}^n \times \mathbb{R}) \longrightarrow \underline{\Omega^{q-1}}(\mathbb{R}^n \times \mathbb{R})$

Main idea : remove "t" , $\omega \in \Omega^q(\mathbb{R}^n \times \mathbb{R})$ $\Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$

(I) there is no dt : $\exists\, \phi \in \Omega^q(\mathbb{R}^n)$

$$\omega = \boxed{(\pi^* \phi)} \cdot f(x,t) \xmapsto{\quad K \quad} 0$$

(II) there is a dt : $\exists\, \phi \in \Omega^{q-1}(\mathbb{R}^n)$

$$\omega = (\pi^* \phi) \wedge \underline{\underline{f(x,t)\,dt}} \xmapsto{\quad K \quad} \underline{\pi^* \phi} \cdot \int_0^t f\,dt$$

The Poincaré lemma for de Rham cohomology

Goal: check such K really satifies $1 - \pi^* \circ s^* = \pm(dK \pm Kd)$

We only check (I).

(I) suppose $\omega = (\pi^*\phi) \cdot f(x,t) \in \Omega^q(\mathbb{R}^n \times \mathbb{R})$

$$\pi^* s^* \omega = \pi^*\left(s^*(\pi^*\phi) \cdot s^* f \right)$$

$$= \pi^*(\phi \cdot (f \circ s)) = (\pi^*\phi) \cdot (f \circ s \circ \pi)$$

$$= \pi^*\phi \cdot f(x,0)$$

$$\Rightarrow (1 - \pi^* s^*)\, \omega = \pi^*\phi \cdot \left(f(x,t) - f(x,0) \right)$$

The Poincaré lemma for de Rham cohomology

$$K\omega = 0 \Rightarrow dK\omega = 0$$

$$d\omega = d(\pi^*\phi \cdot f(x,t))$$

$$K \; = \underbrace{d(\cancel{\pi^*\phi})\cdot f}_{(\mathrm{I})} + \underbrace{\boxed{(-1)^q}\,\cancel{(\pi^*\phi)}\wedge\Big(\frac{\partial f}{\partial x}dx}_{(\mathrm{I})} + \underbrace{\frac{\partial f}{\partial t}dt}_{(\mathrm{II})}\Big)$$

$$Kd\omega = K\left(\pi^*\phi\wedge\frac{\partial f}{\partial t}dt\right) = \pi^*\phi\cdot\int_0^t \frac{\partial f}{\partial t}dt$$

$$= \pi^*\phi\,(f(x,t)-f(x,0)) \Rightarrow 1-\pi^*\circ s^* \quad (\mathrm{I})$$

$$= (-1)^{q-1}(dK-Kd)$$

The Poincaré lemma for de Rham cohomology

Thm $H^*(\mathbb{R}^n) \cong H^*(\mathbb{R}^{n-1}) \cong \cdots \cong H^*(pt)$

Proof: By induction. $H^*(\mathbb{R}^{n+1}) \underset{s^\#}{\overset{\pi^\#}{\rightleftarrows}} H^*(\mathbb{R}^n)$

Corollary: using the same observation:

$$\begin{tikzcd} M \times \mathbb{R}^1 \arrow[d, shift right, "\pi"'] \\ M \arrow[u, shift right, "s"'] \end{tikzcd} \Longrightarrow \begin{tikzcd} H^*(M \times \mathbb{R}^1) \arrow[d, shift left, "s^\#"] \\ H^*(M) \arrow[u, shift left, "\pi^\#"] \end{tikzcd}$$

$s(x) = (x,0)$
$(x,1)$

Proof: locally use K operator.

The Poincaré lemma for de Rham cohomology

Corollary Homotopic maps induce the same map in cohomology.

$x \in M$

Proof: Recall $f \overset{F}{\simeq} g : M \to N$. i.e. $F : M \times \mathbb{R}^1 \to N$

$$\begin{cases} F(x,t) = f(x) & t \geq 1 \\ F(x,t) = g(x) & t \leq 0 \end{cases}, \quad \text{suppose} \begin{cases} S_1(x) = (x,1) \\ S_0(x) = (x,0) \end{cases}$$

$\Rightarrow f = F \circ S_1 \ , \ g = F \circ S_0$

$$\Rightarrow \underbrace{f^{\#}} = S_1^{\#} \circ F^{\#} = (\pi^{\#})^{-1} \circ F^{\#} = S_0^{\#} \circ F^{\#}$$

$\overset{\textstyle \underline{g^{\#}}}{\shortparallel}$

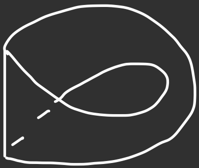
The Poincaré lemma for de Rham cohomology

The case for compact cohomology is similar.

$$H_c^{*+1}(\mathbb{R}^{n+1}) \cong H_c^{*}(\mathbb{R}^{n})$$

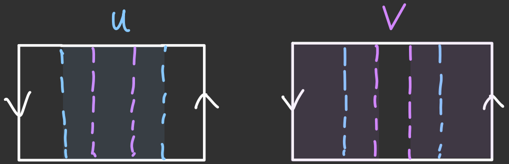
$$H_c^{*+1}(M\times \mathbb{R}) \cong H_c^{*}(M)$$

Exercise 4.8. compute $H^*(M)$. $H_c^*(M)$. M is open Möbius strip.



① $M \simeq S^1 \Rightarrow H^*(M) \cong H^*(S^1)$

②



Mayer-Vietoris Argument

Motivation: The way we calculate $H^*(S^n)$

Main Tools

- Geometric tools: induction on the cardinality of an special open cover —the good cover;
- Algebraic tools: Five Lemma

Given a $\boxed{\text{commutative diagram}}$ of Abealian grps

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "f_1"] \arrow[d, "\alpha"', "\cong"] & B \arrow[r, "f_2"] \arrow[d, "\beta"', "\cong"] & C \arrow[r, "f_3"] \arrow[d, "r"'] & D \arrow[r, "f_4"] \arrow[d, "\delta"', "\cong"] & E \arrow[r] \arrow[d, "r"', "\cong"] & \cdots \\
\cdots \arrow[r] & A' \arrow[r, "f_1'"] & B' \arrow[r, "f_2'"] & C \arrow[r, "f_3'"] & D' \arrow[r, "f_4'"] & E' \arrow[r] & \cdots
\end{tikzcd}$$

then r is also a isomorphism

Main Resluts

- the finite dimensionality of de Rham cohomology;
- Poincaré duality;
- the Künneth formula — compute the cohomology of product space;
- the Leray-Hirsch theorem — compute the cohomology of fiber bundle.

Existence of a Good Cover

Definition (good cover)

Let M be a n manifold, an open cover $\mathcal{U} = \{U_\alpha\}$ of M is called a **good cover** if all nonempty finite intersections $U_{\alpha_0} \cap \cdots U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. A manifold which has a finite good cover is said to be of **finite type**.

Theorem

Every manifold has a good cover. → Rieman mfld

→ Geodesically convex nbhd

① $\cong \mathbb{R}^n$

② ...

Corollary

Ever compact manifold is finite type.

Proof of exisetnce of a good cover

Proof.

Some basic knowledge from Riemannian geometry may be helpful:

1. Every manifold can be given a Riemannian structure, and Levi-Civta connection, thus we can define geodesic;
2. From Whitehead theorem in Rienmannian Geometry, every point has a *geodesically convex nbhd*; [Do carmo P76]
3. If U and V are g.c., then $U \cap V$ is a g.c.n;
4. The geodecically convex nbhd can be choosen sufficiently small s.t. it lies in the coordinate nbhd, then diffeomorphic to $\mathbb{R}^n$.

□

Application I: Finite Dimensionality

the finite dimensionality of de Rham cohomology

$M \cong \mathbb{R}^n$ ✓

Theorem

If M has a finite good cover, then we have each $q \in \mathbb{N}$,

$$\dim_{\mathbb{R}} H^q(M) < \infty.$$

$U \cup V = M$

The most important observation: by MV sequence

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{d^*} \boxed{H^q(U \cup V)} \xrightarrow{r} H^q(U) \oplus H^q(V) \to \cdots,$$

We get

$$H^q(U \cup V) \cong \ker r \oplus \operatorname{im} r = \boxed{\operatorname{im} d^* \oplus \operatorname{im} r}.$$

the finite dimensionality of de Rham cohomology

N $\quad U_0 \cdots U_{p-1}$ $\quad \dim_{\mathbb{R}} H^q(N) < \infty$

Proof.

Induction: the number of covers in the good cover
If M is already diffeomorphic to $\mathbb{R}^n$, then it follows from Poincaré lemma.

$\dim_{\mathbb{R}} H^q(U) < \infty$

Now for $p \to p+1$:
Suppose good cover is $\{U_0, \cdots, U_p\}$, consider $\boxed{U = U_0 \cup \cdots \cup U_{p-1}}$ and $V = U_p$, it follows from the observation above. □

Remark. It is also true for compact cohomology.

$U \cap V$

$= \{U_{0p} \cdots U_{p-1,p}\}$

$U_0 \cap U_p \cdots U_{p-1} \cap U_p$

Application II: Poincaré Duality

Some observations

$$\begin{array}{lcccccc} & 0 & & & & & n \\ H^q(\mathbb{R}^n) & \mathbb{R} & 0 & - & - & - & 0 \\ H_c^q(\mathbb{R}^n) & 0 & 0 & - & - & - & \mathbb{R} \end{array}$$

For $\forall n \in \mathbb{N}$:

$$H^q(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } q = 0 \\ 0 & \text{if } q \geq 1 \end{cases}, \quad H_c^q(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } q = n \\ 0 & \text{if } q \neq n \end{cases}.$$

For sphere S^n:

$$H^q(S^n) = \boxed{H_c^q(S^n)} = \begin{cases} \mathbb{R} & \text{if } q = 0, n \\ 0 & \text{else} \end{cases}.$$

$$\begin{array}{ccccccc} \mathbb{R} & 0 & - & - & - & 0 & \mathbb{R} \\ \mathbb{R} & 0 & - & - & - & 0 & \mathbb{R} \end{array}$$

Basic Algebraic knowledge

A pairing between V. W. $\dim_{\mathbb{R}} V, \dim_{\mathbb{R}} W < \infty$

$$\langle \ , \ \rangle : V \otimes W \to \mathbb{R} \qquad \langle v, w \rangle \in \mathbb{R}$$

is said to be non degenerate. if

$$\langle v.w \rangle = 0. \ \forall w \in W \Rightarrow v = 0$$

i.e the map $v \mapsto \langle v, \cdot \rangle \in W^*$ is injective.

Thm. $\dim_{\mathbb{R}} V, \dim_{\mathbb{R}} W < \infty$. $\langle \ , \ \rangle$ is nondegenerate iff the map $\boxed{v \mapsto \langle v, \cdot \rangle}$ is an isomorphism

i.e. $V \cong W^*$.

Basic Algebraic knowledge

The induced map :

$$\begin{tikzcd} V_1 \arrow[r, "f"] & V_2 \\ \otimes & \otimes \\ W_1 \arrow[d, "\langle , \rangle"'] & W_2 \arrow[l, "g"'] \arrow[d, "\langle , \rangle"] \\ \mathbb{R} & \mathbb{R} \end{tikzcd}$$

v $\quad f(v)$ $\quad g(w)$ $\quad w$

$$\boxed{\langle v, g(w) \rangle = \langle f(v), w \rangle}$$

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "a"'] & V_2 \arrow[d, "b"] \\ W_1^* \arrow[r, "g^*"] & W_2^* \end{tikzcd}$$

$v \mapsto \langle v, \cdot \rangle$ $\quad f(v) \mapsto \langle f(v), \cdot \rangle$

$$bf = g^* a$$

$$bf = -g^* a$$

Poincaré duality

$\langle [\sigma], [\tau] \rangle$

Theorem

Suppose M is oriented n manifold, there is a pairing $:= \int_M \sigma \wedge \tau$

$$\int : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}, \quad [\sigma] \otimes [\tau] \mapsto \int_M \sigma \wedge \tau,$$

If M has a finite good cover, then the pairing is nondegenerate ,i.e.

$$H^q(M) \cong (H_c^{n-q}(M))^*.$$

$u \cdot v$

Check!: Is the pairing well defined? P $\mathbb{R}^n$

How to prove?

Similar to the previous application, we hope the theorem can be proved by induction ,more precisely, if the thm holds for U, V and $U \cap V$, then it holds for $U \cup V$.
From MV seq , we have (note the different direction)

$$\cdots \leftarrow H_c^{n-q}(U\cup V) \leftarrow H_c^{n-q}(U)\oplus H_c^{n-q}(V) \leftarrow H_c^{n-q}(U\cap V) \leftarrow \cdots$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^{n-q}(U\cup V)^* \arrow[r] \arrow[d, phantom, "\wr ??"] & H_c^{n-q}(U)^* \oplus H_c^{n-q}(V)^* \arrow[r] \arrow[d] & H_c^{n-q}(U\cap V)^* \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^q(U\cup V) \arrow[r] & H^q(U)\oplus H^q(V) \arrow[r] & H^q(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Using Five Lemma

Main obstruction :

$H^q(U)$ $H_c^{n-q}(U)$

① how to define the map $\underbrace{H^q(U) \to H_c^{n-q}(U)^*}_{\partial\Delta}$?

induce from $\int \sigma \wedge \tau$

② how to show the commutative ?

$$
\begin{tikzcd}
\cdots \arrow[r] & H^q(U \cup V) \arrow[r, "\text{restriction}"] & H^q(U) \oplus H^q(V) \arrow[r, "\text{difference}"] & H^q(U \cap V) \arrow[r, "d^*"] & H^{q+1}(U \cup V) \arrow[r] & \cdots \\
 & \otimes & \otimes & \otimes & \otimes & \\
\cdots & H_c^{n-q}(U \cup V) \arrow[l] \arrow[d, "\int_{U \cup V}"] & H_c^{n-q}(U) \oplus H_c^{n-q}(V) \arrow[l, "\text{sum}"'] \arrow[d, "\int_U + \int_V"] & H_c^{n-q}(U \cap V) \arrow[l] \arrow[d, "\int_{U \cap V}"] & H_c^{n-q-1}(U \cup V) \arrow[l, "d_*"'] \arrow[d, "\int_{U \cup V}"] & \\
 & \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} &
\end{tikzcd}
$$

Using Five Lemma

$\omega.$ $\quad d^*\omega$

$d_*\tau$ $\quad \tau$

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(U\cup V) \arrow[r, "\text{restriction}"] & H^q(U)\oplus H^q(V) \arrow[r, "\text{difference}"] & H^q(U\cap V) \arrow[r, "d^*"] & H^{q+1}(U\cup V) \arrow[r] & \cdots \\
& \otimes & \otimes & \otimes & \otimes & \\
\cdots & H_c^{n-q}(U\cup V) \arrow[l] \arrow[d, "\int_{U\cup V}"] & H_c^{n-q}(U)\oplus H_c^{n-q}(V) \arrow[l, "\text{sum}"'] \arrow[d, "\int_U + \int_V"] & H_c^{n-q}(U\cap V) \arrow[l] \arrow[d, "\int_{U\cap V}"] & H_c^{n-q-1}(U\cup V) \arrow[l, "d_*"'] \arrow[d, "\int_{U\cup V}"] & \\
& \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} &
\end{tikzcd}$$

check: $\langle \omega, d_*\tau \rangle$ $\overset{=\pm}{\text{and}}$ $\langle d^*\omega, \tau \rangle$

Recall: $d^*\omega|_U = -d(\rho_V\,\omega).\quad d^*\omega|_V = d(\rho_U\,\omega)$

$$d_*\tau = d(\rho_U\tau) = d(\rho_V\tau)$$

The last is direct calculation.

$$\langle \omega, d_*\tau \rangle = \int_{U\cap V} \omega \wedge d_*\tau = (-1)^{\deg\omega+1}\int_{U\cup V} d^*\omega\wedge\tau = (-1)^{\deg\omega+1}\langle d^*\omega, \tau\rangle$$

the induced diagram is sign-commu.

Using Five Lemma

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(U)\oplus H^q(V) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & H^q(U\cap V) \arrow[r, "a"] \arrow[d, "c", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & H^{q+1}(U\cup V) \arrow[r] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & H^{q+1}(U)\oplus H^{q+1}(V) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & H^{q+1}(U\cap V) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_c^{n-q}(U)^*\oplus H_c^{n-q}(V)^* \arrow[r] & H_c^{n-q}(U\cap V)^* \arrow[r, "-d"] & H_c^{n-q-1}(U\cup V)^* \arrow[r, "e"] & H_c^{n-q-1}(U)^*\oplus H_c^{n-q-1}(V)^* \arrow[r] & H_c^{n-q-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

$\operatorname{Im} d \supset$ here

$\circlearrowright$: commutative from $\langle [\sigma], r_*[\tau]\rangle = \langle r^*[\sigma], [\tau]\rangle$ $\operatorname{Im}(d)$

$\blacksquare$: $\int_{U\cap V} \omega\wedge d_*\tau = (-1)^{\deg\omega+1}\int_{U\cup V} d^*\omega\wedge\tau$ $\supset$ here

i.e. $ba = -dc = (-d)c$

How to prove?

Proof.

Induction: the number of covers in the good cover
If M is already diffeomorphic to $\mathbb{R}^n$, then it follows from Poincaré lemma.

Now for $p \to p+1$:
Suppose good cover is $\{U_0, \cdots, U_p\}$, consider $U = U_0 \cup \cdots \cup U_{p-1}$ and $V = U_p$, it follows from the observation above. $\square$

degree of map

Recall: M is $\boxed{\text{finite type}}$ then $\boxed{H^q(M)} \cong (H_c^{n-q}(M))^*$

Corollary

If M is a connected oriented manifold of dimension n, then $(H_c^n(M))^* \cong H^0(M) = \mathbb{R}$, then $H_c^n(M) \cong \mathbb{R}$, if M is also compact, then $H^n(M) = \mathbb{R}$.

If M, N compact, connected, oriented, $f : M \to N$, $f^* : H^n(N) \to H^n(M)$, then the degree of f is

$$\deg f = \int_M f^*\omega,$$

where $H^n(N) = \langle[\omega]\rangle$.

$\triangle \, D$

Application III: Künneth formula and Leray-Hirsch Theorem

$X \times Y$

Künneth formula

Künneth formula mainly tells us how to compute the cohomology of the product of two manifolds M and F, i.e.

$$H^*(M \times F) = H^*(M) \otimes H^*(F),$$

More precisely

$$H^n(M \times F) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(F).$$

To have a better understanding of MV argument this powerful tool, we will assume only M is finite type.

Construct the isomorphiam ψ

The two natural projections

$$\begin{tikzcd} \omega & M \times F \arrow[r, "\rho"] \arrow[d, "\pi"] & F & \phi \\ & M & & \end{tikzcd}$$

give rise to a map on forms

$\psi \quad \rho \quad \pi$

$$\omega \otimes \phi \mapsto \pi^* \omega \wedge \rho^* \phi, \quad \omega \in H^p(M), \phi \in H^q(F),$$

which induces a map in cohomology

$$\psi : H^*(M \times F) = H^*(M) \otimes H^*(F),$$

we will show that ψ is an isomorphism.(Five lemma !)

Using MV argument to prove

Main idea is also :induction + MV sequence, so if $M = \mathbb{R}^n$, it follows from Poincaré lemma.
Now consider MV seq below:

$$\cdots \to H^p(U \cup V) \to H^p(U) \oplus H^p(V) \to H^p(U \cap V) \to \cdots,$$

1. tensoring $H^{n-p}(F)$, still exact;
2. summing over all integers p.

Using Five Lemma

we get an exact sequence by tensoring with $\boxed{H^{n-p}(F)}$

$$\cdots \to H^p(U \cup V) \otimes H^{n-p}(F) \to (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F))$$
$$\to H^p(U \cap V) \otimes H^{n-p}(F) \to \cdots$$

since tensoring with a vector space preserves exactness. Summing over all integers p yields the exact sequence

$$\cdots \to \bigoplus_{p=0}^{n} H^p(U \cup V) \otimes H^{n-p}(F)$$
$$\to \bigoplus_{p=0}^{n} (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F))$$
$$\to \bigoplus_{p=0}^{n} H^p(U \cap V) \otimes H^{n-p}(F) \to \cdots.$$

$U \cap V$. –

The following diagram is commutative

$$\begin{tikzcd}
\bigoplus_{p=0}^{n} H^p(U \cup V) \otimes H^{n-p}(F) \arrow[r] \arrow[d, "\psi"] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_{p=0}^{n} (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \arrow[r] \arrow[d, "\psi"] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_{p=0}^{n} H^p(U \cap V) \otimes H^{n-p}(F) \arrow[d, "\psi"] \\
H^n((U \cup V) \times F) \arrow[r] & H^n(U \times F) \oplus H^n(V \times F) \arrow[r] & H^n((U \cap V) \times F)
\end{tikzcd}$$

$M \times F$

Leray-Hirsch Theorem

Let $\pi : E \to M$ be a fiber bundle with fiber F, suppose there are cohomology classes $e_1, \cdots, e_r$ on E which restrict to a basis of the cohomology of each fiber. Then we can define a map

$$\psi : H^*(M) \otimes \mathbb{R}\{e_1, \cdots, e_r\} \to H^*(E).$$

The same argument as the Künneth formula gives

Theorem

Let E be a fiber bundle over M with fiber F, M is finite type, if there are cohomology classes $e_1, \cdots, e_r$ on E which restrict to a basis of the cohomology of each fiber, then

$$H^*(E) \cong H^*(M) \otimes \mathbb{R}\{e_1, \cdots, e_r\} \cong H^*(M) \otimes H^*(F).$$

The End