2023 Fall Partial Differential Equations Midterm Task: Properties of Harmonic Functions

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October 29, 2023

Contents

1 Problem 9.1

Problem. *Suppose that* $u, \phi \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega})$ *on a bounded domain* $\Omega \subset \mathbb{R}^n$*. Assume that* u *subharmonic and* ϕ *harmonic, with matching boundary values:*

$$u|_{\partial\Omega} = \phi|_{\partial\Omega}.$$

Show that

$$u \leq \phi$$

at all points of Ω*.*

Proof. Let $v = u - \phi \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega})$, then we have

$$-\Delta v = -\Delta u + \Delta \phi = -\Delta u \leq 0.$$

Hence v is subharmonic, and

$$v|_{\partial\Omega} = u|_{\partial\Omega} - \phi|_{\partial\Omega} = 0.$$

Then from maximum principle, we have

$$v \leq 0$$

at all points of Ω. We can now conclude that $u \leq \phi$ at all points of Ω. □

2 Problem 9.2

Problem. *Liouville's theorem says that a bounded harmonic function on* $\mathbb{R}^n$ *is constant. To show this, assume* $u \in C^2(\mathbb{R}^n)$ *is harmonic and satisfies*

$$|u(\boldsymbol{x})| \leq M$$

for all $\boldsymbol{x} \in \mathbb{R}^n$*.*

Proof. For arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^n$, set $r_0 = |\boldsymbol{x}_0|$. From mean value theorem, we have

$$u(0) = \frac{1}{\mathrm{Vol}[B(0,R)]} \int_{B(0,R)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$
$$u(\boldsymbol{x_0}) = \frac{1}{\mathrm{Vol}[B(\boldsymbol{x}_0,R)]} \int_{B(\boldsymbol{x}_0,R)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Hence we have

$$u(0) - u(\boldsymbol{x}_0) = \frac{n}{A_n R^n}\left[\int_{B(0,R)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{B(\boldsymbol{x}_0,R)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right].$$

Let $U = B(0,R) \setminus B(\boldsymbol{x}_0,R)$ and $V = B(\boldsymbol{x}_0,R) \setminus B(0,R)$. Since the two domians are symmetric, we have $\mathrm{Vol}(U) = \mathrm{Vol}(V)$. Note that $B(0,R) \setminus B(\boldsymbol{x}_0,R) \subset B(0,R) \setminus B\left(\frac{\boldsymbol{x}_0}{2}, R - \frac{r_0}{2}\right)$, then we have

$$\begin{aligned}
|u(0) - u(\boldsymbol{x}_0)| &\leq \frac{n}{A_n R^n}\left[\int_U u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \int_V u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right], \\
&\leq \frac{n}{A_n R^n} \cdot M \cdot (\mathrm{Vol}(U) + \mathrm{Vol}(V)) \\
&= \frac{2nM}{A_n R^n} \cdot \mathrm{Vol}(U) \\
&\leq \frac{2nM}{A_n R^n} \cdot \mathrm{Vol}\left(B\left(\frac{\boldsymbol{x}_0}{2}, R - \frac{r_0}{2}\right)\right) \\
&\leq 2M\left[\frac{R^n - (R - \frac{r_0}{2})^n}{R^n}\right].
\end{aligned}$$

Take $R \to \infty$, then we have $u(\boldsymbol{x}_0) = u(0)$. Hence u is constant. $\square$

3 Problem 9.3

Problem. *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded, with $\Omega \subset B(0,R)$, and assume that $u \in C^2(\Omega;\mathbb{R}) \cap C^0(\overline{\Omega})$ satisfies*

$$-\Delta u = f, \quad u|_{\partial\Omega} = 0$$

and $f \in C^0(\overline{\Omega})$.Show that there exists a constant C depends only on R such that

$$\max_{\overline{\Omega}} |u| \leq C \max_{\overline{\Omega}} |f|.$$

Proof. Let $M = \max_{\overline{\Omega}} |f|$, and $c = \frac{M}{2n}$. Consider $g(\boldsymbol{x}) = u(\boldsymbol{x}) + c|\boldsymbol{x}|^2$, thus

$$-\Delta g = -\Delta u - 2nc = f - M \leq 0.$$

Hence g is subharmoinc. Note that for any $\boldsymbol{x} \in \partial\Omega$, we have

$$|g(\boldsymbol{x})| = c|\boldsymbol{x}|^2 \leq cR^2.$$

Thus $\max_{\partial\Omega} |g| \leq cR^2$, then from maximum principle,

$$\max_{\overline{\Omega}} |g| \leq \max_{\partial\Omega} |g| \leq cR^2.$$

Note that $|u(\boldsymbol{x})| \leq |g(\boldsymbol{x})| + cR^2$, then we have

$$\max_{\overline{\Omega}} |u| \leq \max_{\overline{\Omega}} |g| + cR^2 \leq 2cR^2 = \frac{R^2}{n} \cdot \max_{\overline{\Omega}} |f|.$$

Hence we choose $C = \frac{R^2}{n}$ then complete the proof. $\square$

4 Problem 9.4

Problem. *Suppose u is a harmonic function on a domain that includes $B(0,4R)$ for some $R>0$, and assume $u \geq 0$. Show that*

$$\max_{B(0,R)} u \leq 3^n \min_{B(0,R)} u.$$

Proof. Asuume $u(\boldsymbol{x}^*) = \max_{B(0,R)} u$, $u(\boldsymbol{x}_*) = \min_{B(0,R)} u$. From mean value theorem, we have

$$u(\boldsymbol{x}^*) = \frac{1}{\mathrm{Vol}[B(\boldsymbol{x}^*,R)]} \int_{B(\boldsymbol{x}^*,R)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$
$$u(\boldsymbol{x}_*) = \frac{1}{\mathrm{Vol}[B(\boldsymbol{x}_*,3R)]} \int_{B(\boldsymbol{x}_*,3R)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

From the figure below, we can easily have $B(\boldsymbol{x}^*,R) \subset B(0,2R) \subset B(\boldsymbol{x}_*,3R)$.

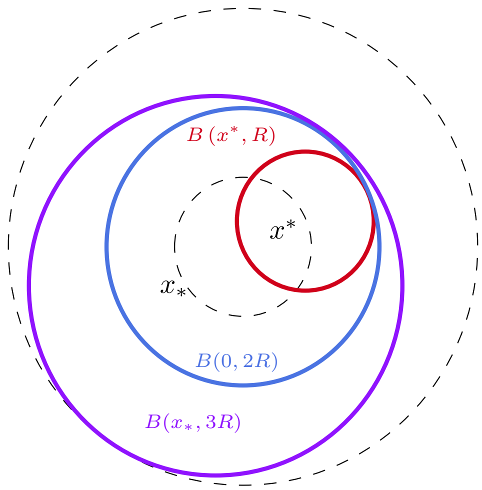

Figure 1: $B(\boldsymbol{x}^*,R) \subset B(0,2R) \subset B(\boldsymbol{x}_*,3R)$

Hence we have

$$\begin{aligned} u(\boldsymbol{x}_*) &= \frac{n}{A_n(3R)^n} \int_{B(\boldsymbol{x}_*,3R)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &\geq \frac{n}{A_n(3R)^n} \int_{B(\boldsymbol{x}^*,R)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &= \frac{n}{A_n(3R)^n} \cdot \frac{A_n R^n}{n} \cdot u(\boldsymbol{x}^*). \end{aligned}$$

Thus $u(\boldsymbol{x}^*) \leq 3^n u(\boldsymbol{x}_*)$, then we know that $\max_{B(0,R)} u \leq 3^n \min_{B(0,R)} u$. □

5 Problem 9.5

Problem. *Suppose $u \in C^2(B,\mathbb{R}) \cap C^0(\overline{B})$ is a nonconstant subharmonic function and assume that the maximum of u on $\overline{B}$ is attained at the point $\boldsymbol{x}_0 \in \partial B$. Prove Hopf's lemma, i.e.,*

$$\frac{\partial u}{\partial r}(\boldsymbol{x}_0) > 0.$$

Proof. To show this, let $B := B(0,R) \subset \mathbb{R}^n$ for some $R > 0$, and set

$$A := \left\{ \frac{R}{2} < |\boldsymbol{x}| < R \right\}.$$

Consider the function

$$h(\boldsymbol{x}) := \mathrm{e}^{-2n|\boldsymbol{x}|^2/R^2} - \mathrm{e}^{-2n}.$$

Then we have

$$\frac{\partial h}{\partial x_j}(\boldsymbol{x}) = \mathrm{e}^{-2n|\boldsymbol{x}|^2/R^2} \cdot \frac{-2n}{R^2} \cdot 2x_j,$$
$$\frac{\partial^2 h}{\partial x_j^2}(\boldsymbol{x}) = \mathrm{e}^{-2n|\boldsymbol{x}|^2/R^2} \left(\frac{16n^2 x_j^2}{R^4} - \frac{4n}{R^2} \right).$$

Hence we have

$$-\Delta h = -\mathrm{e}^{-2n|\boldsymbol{x}|^2/R^2} \left(\frac{16n^2|\boldsymbol{x}|^2}{R^4} - \frac{4n^2}{R^2} \right) \leq 0$$

for any $\boldsymbol{x} \in A$. Now set $m = \max_{\{r=\frac{R}{2}\}} u$ and $M = \max_{\{r=R\}} u$. If $m \geq M$, then there exists $|\boldsymbol{x}_0^*| = \frac{R}{2}$ such that $u(\boldsymbol{x}_0^*) = \max_{\overline{B}} u$. Since $\boldsymbol{x}_0^* \in B$, we can attain a contracdiction from strong manimum principle. Thus $m < M$.

Hence we can choose $0 < \varepsilon < \frac{M-m}{2(\mathrm{e}^{-n/2} - \mathrm{e}^{-2n})}$. Consider

$$u_\varepsilon := u + \varepsilon h.$$

Then we have

$$u_\varepsilon|_{\partial B} = u|_{\partial B} \leq M$$

and

$$\begin{aligned} u_\varepsilon|_{\partial B(0,R/2)} &= u|_{\partial B(0,R/2)} + \varepsilon h|_{\partial B(0,R/2)} \\ &\leq m + \varepsilon \cdot (\mathrm{e}^{-n/2} - \mathrm{e}^{-2n}) < M. \end{aligned}$$

Thus we have

$$\max_{\partial A} u_\varepsilon \leq M.$$

Since $-\Delta u_\varepsilon = -\Delta u - \varepsilon \Delta h \leq 0$ on A, from maximum principle, we have

$$\max_{\overline{A}} u_\varepsilon = \max_{\partial A} u_\varepsilon \leq M.$$

Thus we have

$$\frac{\partial u_\varepsilon}{\partial r}(\boldsymbol{x}_0) = \lim_{h \to 0^+} \frac{u_\varepsilon(\boldsymbol{x}_0) - u_\varepsilon(\boldsymbol{x}_0 - hr)}{h} = \lim_{h \to 0^+} \frac{M - u_\varepsilon(\boldsymbol{x}_0 - hr)}{h} \geq 0.$$

Note that

$$\frac{\partial h}{\partial r}(\boldsymbol{x}_0) = \nabla h(\boldsymbol{x}_0) \cdot \boldsymbol{x_0} = -4n\mathrm{e}^{-2n},$$

then we have

$$\frac{\partial u}{\partial r}(\boldsymbol{x}_0) = \frac{\partial u_\varepsilon}{\partial r}(\boldsymbol{x}_0) - \varepsilon \frac{\partial h}{\partial r}(\boldsymbol{x}_0) \geq 4n\varepsilon \mathrm{e}^{-2n} > 0.$$

Finally we complete the proof. □

References

[1] D. Borthwick, *Introduction to partial differential equations.* Springer, 2017.