Solutions to 2023 Differentiable Mnifolds Homework

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October 25, 2023

Abstract

These homeworks are from the course website, the course is taught by Prof. Zuoqing Wang.

Contents

1 Problem Set 1, Part 1: Smooth Manifolds

Problem 1.1 (Topological Group). *A topological group is a topological space G which is also a group, so that the multiplication operation*

$$\mu : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 g_2$$

and the inverse operation

$$i : G \to G, \quad g \mapsto g^{-1}$$

are both continuous. For any subsets S and T in G we can define

$$ST = \{g_1 g_2 | g_1 \in S, g_2 \in T\}$$

By this way we can define the subsets $S^n = S \times \cdots \times S$ and S^{-1}.

1. *If G is a topological group, and U is any open neighborhood of the identity element $e \in G$. Prove: There exists an open neighborhood V of e so that $V = V^{-1}$ and $V^2 \subset U$.*

2. *Prove: If G is a connected topological group, then for any open neighborhood U of the identity element $e \in G$, we have*

$$G = \bigcup_{n=1}^{\infty} U^n.$$

3. *Suppose G is compact Hausdorff topological group and $g \in G$ Prove: $e \in \overline{\{g^n | n \in \mathbb{Z}^*\}}$.*

Proof. (1) Note that i is a homeomorphism, and $V^{-1} = i(V)$, thus for each open subset V, we have V^{-1} is open, so $V \cap V^{-1}$ is open, then for any U is an open neighborhood of e, then we have $U \cap U^{-1}$ is an open neighborhood of e, and $(U \cap U^{-1})^{-1} = U \cap U^{-1}$.

Note that $V^2 = \mu(V \times V)$, thus since μ is continuous, and $e \cdot e = e$, thus for the open neighborhood U of e, then we can always find a neighborhood of (e, e) in $G \times G$, more precisely, we can choose $V \times V$ such that $V \times V \subset \mu^{-1}(U)$, i.e., we have $V^2 \subset U$, then from $A \subset B$ then $A^2 \subset B^2$, we can choose $V \cap V^{-1}$ as desired.

(2) Firstly we claim an important fact: open subgroup of topological group is always closed. Suppose V is an open subgroup of G, then we have for any $g \in G$, gV is open, then we have

$$V = G \setminus \bigcup_{g \neq e} gV$$

is closed. Now consider $V \subseteq U$ and $V = V^{-1}$, then we have

$$H = \bigcup_{n=1}^{\infty} V^n$$

is natural a subgroup of G, note that $V^2 = \cup_{g \in V} gV$ is open, so we can prove V^n is open for all n by induction, then we konw that H is an open subgroup of G, then it is closed, since G is connected, then we know that $H = G$, since $H \subseteq \cup U^n \subseteq G$, then we finish the proof.

(3) □

Problem 1.2 (Locally Euclidean)**.** *Prove the following properties of locally Euclidean spaces:*

1. *Any connected component of a locally Euclidean space is open.*
2. *Any connected locally Euclidean space is path connected.*
3. *Any locally Euclidean Hausdorff space is regular. Thus as a consequence of Urysohn's metrization theorem, any topological manifold is metrizable.*
4. *If both X, Y are connected, second countable and locally Euclidean, and $f : X \to Y$ is bijective and continuous, then f is a homeomorphism.*

Proof. (1) Recall a basic fact: if A and B are connected and $A \cap B \neq \varnothing$ then $A \cup B$ is connected, thus for a connected component C of X, and any $x \in C$, since X is locally Euclidean, then there exists a coordinate chart (x, U, φ) of x, then since U is homeomorphism to $\mathbb{R}^n$ through φ, then we know that U is connected, then since $x \in C \cap U$, thus we have $C \cup U$ is connected, then we have $C \cup U = C$, then $U \subset C$, then we know that C is open.

(2) For any $x \in X$, we define

$$\mathscr{P}_x := \{y \in X | \exists \gamma : I \to X, \gamma(0) = x, \gamma(1) = y\},$$

then we know that $\mathscr{P}_x = \mathscr{P}_y$ or $\mathscr{P}_x \cap \mathscr{P}_y = \varnothing$, from the locally Euclidean condition, we know that $\mathscr{P}_x$ is open for all $x \in X$, then since

$$X = \bigsqcup_{x \in X} \mathscr{P}_x,$$

and from X is connected, we know that $\mathscr{P}_x$ are all equal, so we know that X is path connected.

(3) Recall regular means we can separate point and closed set with open sets, and locally Euclidean implies that X is locally compact, thus we will prove locally compact Hausdorff space is T_3, then regular. Consider $x \notin F$, and F is closed, then consider compact set $K \subseteq X \setminus F$, and $x \in K^\circ$, then K° and $X \setminus K$ is as desired, since K is a compact space in Hausdorff space, then it is closed.

(4) Without proof, we state a strong result in algebraic topology, called **invariance of domain** proved by Brouwer:

Theorem 1. *If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is an injective continuous map, then $f(U)$ and U are homeomorphism given by f.*

then for each x suppose it has coordinate chart (U_x, φ_x), and for Y we have (V_y, ψ_y), then for any open set U, we have $f(U) = f(\cup_{x \in U} U_x) = \cup_{x \in U} f(U_x)$ by shrinking U_x, then since $\varphi_x^{-1}(U_x)$ is open in $\mathbb{R}^n$ and $\phi_{f(x)} \circ f \circ \varphi_x^{-1}$ is an injective continuous map, so from the theorem above, we have $\phi_{f(x)} \circ f \circ \varphi_x^{-1}$ is open, since φ_x and $\psi_{f(x)}$ are homeomorphisms, then we know that f is open, so we have f is a homeomorphism. □

Remark 2. *From (4) we have for any bijective continuous map between topological manifolds, it is naturally a homeomorphism. For the proof of theorem above, one can refer the lecture notes of Prof. Z.Q. Wang.*

Problem 1.3 (Topological manifolds with boundary)**.**

1. *Find the definition of topological manifolds with boundary from literature.*
2. *Prove: If M is a topological n-manifold with boundary, then its boundary, ∂M, is a topological $(n-1)$-manifold without boundary.*
3. *Prove: the product of two topological manifolds with boundaries is a topological manifold with boundary. What is its boundary?*

Proof. (1) An **$n-$manifold with a boundary** is a second countable Hausdorff space in which any point has a neighborhood which is homeomorphic either to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n = \{x^n \geq 0 | (x^1, \cdots, x^n) \in \mathbb{R}^n\}$ endowed with a Euclidean topology.

(2) We define $\partial M := \{x \in M : \text{there is no } (U, \varphi) \text{ of } x \text{ such that } \varphi(U) \cong \mathbb{R}^n\}$. Now we recall a basic fact: the subset of $T_2(C_2)$ space is still $T_2(C_2)$. Thus ∂M is T_2 and C_2.

We will show that ∂M is locally homeomorphic to $\mathbb{R}^{n-1}$. If $x \in \partial M$, then there exists (U, φ) of x such that $\varphi(U) \cong \mathbb{H}^n$. More precisely, $\varphi(x) \in \partial \mathbb{H}^n$. Now we choose $U' = \varphi^{-1}(\varphi(U) \cap \partial \mathbb{H}^n)$. Since $U' = U \cap \partial M$, we know U' is open in ∂M. Hence $(U', \varphi|_{U'})$ is a chart of $x \in \partial M$. And $\varphi(U')$ is homeomorphic to an open set of $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$.

In summary, we have proved that ∂M is a topological (n-1)-manifold without boundary.

(3) Topologically, we have $\partial(M \times N) = \partial(M) \times N \cup M \times \partial N$. But I am confused with the topological boundary and manifold boundary. ☐

Problem 1.4 (Connected topological manifolds are homogeneous)**.** *Let M be a connected topological manifold. Prove: for any $p, q \in M$, there exists a homeomorphism $\varphi : M \to M$ so that $\varphi(p) = q$.*

Proof. We will prove in three steps:

Step 1: $\mathbb{R}^n$ is homogeneous, for any $p, q \in \mathbb{R}^n$, we have $\varphi(x) := x + q - p$ is a natural homeomorphism, and sends p to q.

Step 2: Open n-ball $\mathbb{D}^n$ is homogeneous, naturally, we have a homeomorphism $f : \mathbb{R}^n \to \mathbb{D}^n$, which is given by

$$f(x) = \frac{x}{\sqrt{1 + |x|^2}},$$

and the inverse map is given by

$$g(y) = \frac{y}{\sqrt{1 - |y|^2}}.$$

Thus for any $p, q \in \mathbb{D}^n$, we can construct the homeomorphism ψ as

$$\psi(y) = \left(\frac{y}{\sqrt{1-|y|^2}} - \frac{p}{\sqrt{1-|p|^2}} + \frac{q}{\sqrt{1-|q|^2}} \right) \Bigg/ \sqrt{1 + \left| \frac{y}{\sqrt{1-|y|^2}} - \frac{p}{\sqrt{1-|p|^2}} + \frac{q}{\sqrt{1-|q|^2}} \right|^2}.$$

Since $\psi(p) = q$, and for $y_0 \in \partial \mathbb{D}^n$, we have $\psi(y_0) = \lim_{y \to y_0} \psi(y) = y_0$, thus $\psi|_{\partial \mathbb{D}^n} = \mathrm{id}|_{\partial \mathbb{D}^n}$.

Step 3: For a general topological manifold M, we fix a point p. Then there exists (U, φ) such that $\varphi(U) = \mathbb{D}^n$ and is homeomorphism to U. Then we construct $f : M \to M$ such that $f|_{M \setminus U} = \mathrm{id}|_{M \setminus U}$, and $f|_U = \varphi^{-1} \circ \psi \circ \varphi$, thus we have f is a homeomorphism of U and sends p to a choosen point. Since $f|_{\bar{U}}$ is well defined, and $f|_{\partial U} = \mathrm{id}$, thus we have f gives a homeomorphism. Now since M is connected then path connected, consider the path from p to q, then we will have $f_1, \cdots, f_k$ are homeomorphisms. More precisely, we will have $f_1(p) = p_1, \cdots, f_k(p_{k-1}) = p_k = q$, thus $f_k \circ \cdots \circ f_1$ is the desired homeomorphism.

Finally, we show that any topological manifold is homogeneous. ☐

Problem 1.5 (Local homeomorphism)**.** *Let X, Y be topological spaces. A map $f : X \to Y$ is called a local homeomorphism if for every point $x \in X$, there exists an open set U containing x such that the image $f(U)$ is open in Y, and the restriction $f|_U : U \to f(U)$ is a homeomorphism (with respect to the respective subspace topologies).*

1. *Show that every local homeomorphism is an open map (i.e. maps each open set to an open set).*
2. *Show that if a local homeomorphism is bijective, then it is a homeomorphism.*
3. *Show that if Y is locally Euclidean and $f : X \to Y$ is a local homeomorphism, then X is locally Euclidean.*
4. *Show that if X is locally Euclidean and $f : X \to Y$ is a surjective local homeomorphism, then Y is locally Euclidean.*

Proof. (1) Fixed an open subset U of X. Then for any $x \in U$, we can find a open neighborhood $U_x \subseteq U$ of x, which is also homeomorphism to $f(U_x)$. Now we have

$$f(U) = f\left(\bigcup_{x \in U} U_x \right) = \bigcup_{x \in U} f(U_x).$$

Since $f|_{U_x}$ is homeomorphism, $f|_{U_x}$ is open. Hence $f(U_x)$ are open in Y for all $x \in U$. Thus $f(U)$ is open in Y.

(2) From (1), a local homeomorphism ia an open map. Thus f^{-1} is continuous, which implies that f is homeomorphism.

(3)(4) They are directly from definition. ☐

2 Problem Set 1, Part 2: Smooth Manifolds/Functions/Maps

Problem 2.1 (Construct smooth manifolds by gluing Euclidean open sets)**.** *Let M be a smooth manifold with atlas $\mathscr{A} = \{(\varphi_\alpha, U_\alpha, V_\alpha)\}$,*

1. *Prove: the transition maps $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfy the **cocycle conditions**:*
   (a) $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ *on* $\varphi_\alpha(U\alpha \cap U_\beta \cap U_\gamma)$*;*
   (b) $\varphi_{\alpha\alpha} = \mathrm{Id}_{V_\alpha}$*;*
   (c) $\varphi_{\alpha\beta} = (\varphi_{\beta\alpha})^{-1}$.
2. *Now on the disjoint union $\widetilde{M} := \bigsqcup_\alpha V_\alpha$ we define an equivalence relation via*

$$x \sim y \iff \exists \alpha, \beta \text{ s.t. } x \in V_\alpha, y \in V_\beta \text{ and } y = \varphi_{\alpha\beta}(x).$$

   (a) *Check: $\sim$ is an equivalence relation on $\widetilde{M}$.*
   (b) *Prove: the quotient $\widetilde{M}/\sim$ is homeomorphic to M .*
   (c) *Define a natural smooth structure on $\widetilde{M}/\sim$*

Proof. (1) This is trivial from definitions of $\varphi_{\alpha\beta}$.
(2) □

Problem 2.2 (Orientability of smooth manifolds)**.** *Prove: M, N are orientable if and only if $M \times N$ are orientable.*

Proof. Recall a smooth manifold M of dimension n is orientable if and only if it has a global non-vanishing $n-$form. Suppose $M \overset{i_1}{\to} M \times N$, $N \overset{i_2}{\to} M \times N$ and $M \times N \overset{\pi_1}{\to} M$, $M \times N \overset{\pi_2}{\to} N$. Hence if (M, ω) and (N, η) are orientable, then $\pi_1^*\omega \wedge \pi_2^*\eta$ is a global non-vanishing top form. And if $(M \times N, \alpha)$ is orientable, then locally at $M \times \{q\}$, where $i_1(p) = (p, q)$, we consider $\omega_p = \alpha_{(p,q)}(\partial_{y^1}|_q, \cdots, \partial_{y^n}|_q)$, then one can easily check ω_p is an non-vanishing m-form on $M \times \{q\}$. Thus M is orientable, so does N. □

References